ASPECTS OF POSITIVE LINEAR MAPS AND APPLICATIONS

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

杨宇

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Summary

This thesis mainly concerns the interplay between positive linear maps and quantum entanglement. In the first part of the thesis, after the preliminary Chapter 1 setting up notations and summarising important known results, we study in Chapter 2 certain aspects of positive linear maps between matrix algebras. In particular, we present a decomposition theorem for k-positive linear maps, where $k \geq 2$:

Theorem 2.2.2 *Let ϕ be a non-zero k-positive $(2 \leq k < n \wedge m)$ map in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$. Then there exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely positive map and γ is a p-trivial lifting of a $(k-1)$-positive map in $B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$, for some $p \in \{1, ..., n\}$.*

As a consequence, it gives an affirmative answer to the open problem that every 3×3 positive partial transpose entangled quantum state (PPTES) has Schmidt number 2. Some related implications and further problems are also considered.

In the second part of the thesis, we explore various aspects of the important notion of the Schmidt number of quantum states. In Chapter 3, we use the technique of local projections on tensor product to show that there exist bipartite partial transpose entangled quantum states (PPTES) of any prescribed Schmidt number:

Theorem 3.2.17 *Given any positive integer r, there exist positive integers M, N and a bipartite PPT entangled state $\rho \in M_M(\mathbb{C}) \otimes M_N(\mathbb{C})$ of Schmidt number r.*

Equivalently, in the language of positive maps:

Theorem 3.2.18 *Give any positive integer r, there exist positive integers M, N and an indecomposable map $\phi \in B(M_M(\mathbb{C}), M_N(\mathbb{C}))$ which is r-positive but not $(r+1)$-positive.*

We further construct the notion of joint Schmidt number for multipartite states, and explore its relation with the Schmidt number of bipartite reduced density operators.

Some further discussions concerning the related distillability problem are included in the final Chapter 4.

Contents

Chapter 1

Introduction

In this chapter, some concepts in operator algebra and quantum information are introduced to make this note self-contained. More specifically, necessary ingredients to explain the interplay between positive linear maps and Schmidt numbers of a quantum state are established. The positive linear maps originates from theory of C^*-algebra while the Schmidt number is an important quantum measure related to the separability problem. We shall also mention the search for positive partial transpose entangled states.

1.1 Various Notions of Positive Maps

The study of positive linear maps on C^*-algebras dates back to mid-1950s with Kadison's generalized Schwarz inequality and characterizations of isometries of C^*-algebras [Ka51, Ka52]. Later Stinespring introduced completely positive maps and his famous dilation theorem [Sti55]. Tomiyama further developed some of the basic results on conditional expectations [To57]. It remains a rather specific area within operator algebras until the change came in the 1990's when the dual cone relation between positive linear maps and quantum entanglement is established in a series of papers [St86, It86, EoKy00]. Since then, accelerated by motivations and problems from quantum information theory, the theory of positive linear maps has attracted an increasing interest, as has the development of the theory by both mathematicians and physicists.

For normed spaces X and Y, denote by $B(X,Y)$ the space of all bounded linear operators from X to Y, and simply write $B(X)$ for $B(X,X)$. In particular, let $\mathcal{H}$ be a complex Hilbert space, and denote by $B(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. We shall use $M_{n,m}(\mathbb{C})$ to denote the set of all $n \times m$ matrices with complex entries. We adopt the convention that $M_n(\mathbb{C})$ denotes the set of all $n \times n$ matrices with complex entries. We denote by E_{ij} the canonical matrix unit whose (i,j) entry is 1 and 0 elsewhere. For finite dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with $\dim \mathcal{H} = n$ and $\dim \mathcal{K} = m$ we often identify $B(\mathcal{K}, \mathcal{H})$ with $M_{n,m}(\mathbb{C})$. For an element $\rho \in M_n(\mathbb{C})$, we shall denote by ρ^t the transpose of ρ and denote by ρ^* the conjugate transpose of ρ, respectively. The notion Tr_n denotes the trace on $M_n(\mathbb{C})$. We denote by id_n and τ_n the identity map and the transpose map on $M_n(\mathbb{C})$, respectively.

More generally if $\mathbf{A}$ is a C^*-algebra, $M_n(\mathbf{A})$ denotes the set of all $n \times n$ matrices with entries in $\mathbf{A}$. Note that $M_n(\mathbf{A})$ is also a C^*-algebra. Throughout this note, we will confine ourselves to the usual matrix algebra setting most of the time but it takes no extra effort to define the positivity in the C^*-algebra setting.

Definition 1.1.1 *Let $\mathbf{A}$ and $\mathbf{B}$ be C^*-algebras. A linear map $\phi : \mathbf{A} \to \mathbf{B}$ is said to be positive if $\phi(\mathbf{A}^+) \subseteq \mathbf{B}^+$, where $\mathbf{A}^+$ and $\mathbf{B}^+$ are the convex cones of all positive elements in $\mathbf{A}$ and $\mathbf{B}$, respectively.*

We shall write $\phi \geq 0$ if it is a positive linear map. Although the notion of positive maps is straightforward, it is generically hard to determine whether a linear map, even between low dimensional matrix algebras, is positive. Dedicated examples of positive maps between low dimensional matrix algebras are [Ro85, CKL92]. Given a natural number k, two generalizations of positive maps are k-positive maps and k-copostive maps defined as below.

Definition 1.1.2 *Let $\phi : \mathbf{A} \to \mathbf{B}$ be a linear map between C^*-algebras.*

(a) ϕ is said to be k-positive if $id_k \otimes \phi : M_k(\mathbf{A}) \to M_k(\mathbf{B})$ is positive.

(b) ϕ is said to be k-copositive if $\tau_k \otimes \phi : M_k(\mathbf{A}) \to M_k(\mathbf{B})$ is positive.

By definition one observes that $(k+1)$-positivity (resp. $(k+1)$-copositivity) implies k-positivity (resp. k-copositivity). The notion of complete positivity and complete copositivity are introduced naturally as follows.

Definition 1.1.3 *Let $\phi : \mathbf{A} \to \mathbf{B}$ be a linear map between C^*-algebras.*

(a) ϕ is said to be completely positive if ϕ is k-positive for all $k \in \mathbb{N}$.

(b) ϕ is said to be completely copositive if ϕ is k-copositive for all $k \in \mathbb{N}$.

It relies on the underlying C^*-algebra whether every positive map is completely positive. Generally speaking, completely positive maps are more special than positive maps.

For example, the nontrivial($n \geq 2$) transpose map τ_n between matrix algebras $M_n(\mathbb{C})$ is a positive but not 2-positive map. Meanwhile, we have an affirmative answer in a degenerate case [St13].

Theorem 1.1.1 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* C^**-algebras and either* $\mathbf{A}$ *or* $\mathbf{B}$ *is abelian. Then every positive map* $\phi : \mathbf{A} \to \mathbf{B}$ *is completely positive (or completely copositive).*

Proof. Refer to Theorem 1.2.4, Theorem 1.2.5 and Remark 1.2.6 in chapter 1 of [St13]. □

Let $B^+(M_n(\mathbb{C}), M_m(\mathbb{C}))$ be the convex cone consisting all positive linear maps in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$. An extremal map ϕ in $B^+(M_n(\mathbb{C}), M_m(\mathbb{C}))$ is a map satisfying the condition that $\{\psi \in B^+(M_n(\mathbb{C}), M_m(\mathbb{C})) : \phi \geq \psi\} = \{\lambda\phi : 0 \leq \lambda \leq 1\}$. It is an open problem to determine all the extremal maps of the set $B^+(M_n(\mathbb{C}), M_m(\mathbb{C}))$ except the result by Stømer which classifies all extremal maps of the cone consisting positive maps in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ [St63]. Another notable result by Woronowicz considers the decomposition of positive maps in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ ($mn \leq 6$) as the sum of a completely positive map and a completely copositive map [Wo76]. On the other hand, there are few structural results in the literature concerning the classification of extremal maps in the cone consisting all positive maps in $B(B(\mathcal{H}), B(\mathcal{K}))$, even when $\dim(\mathcal{H}) = \dim(\mathcal{K}) = 3$. Since the set of extremal maps in $B^+(M_n(\mathbb{C}), M_m(\mathbb{C}))$ when $m, n \geq 3$ is unknown, we are not able analyze the positive cone through extremal maps. Meanwhile, following this spirit one may be able to decompose a positive map into the sum of some more "fundamental" maps. Below we summarize the related terminologies.

Definition 1.1.4 *Let* $\phi : \mathbf{A} \to \mathbf{B}$ *be a linear map between* C^**-algebras.*

(a) ϕ *is said to be* (k, l)*-decomposable if it is the sum of a* k*-positive map and a* l*-copositive map.*

(b) ϕ *is said to be decomposable if it is the sum of a completely positive map and a completely copositive map. Otherwise* ϕ *is said to be indecomposable.*

(c) Especially, ϕ *is said to be atomic if it is not* $(2, 2)$*-decomposable.*

One important result concerning complete positivity (resp. complete copositivity) is the Stinespring Theorem, which extends the Gelfand-Naimark-Segal construction [Co97, chapter VIII].

Theorem 1.1.2 (Stinespring's dilation theorem) *Let* $\mathbf{A}$ *be a unital* C^**-algebra and* $\phi : \mathbf{A} \to B(\mathcal{H})$ *a linear map.*

(i) ϕ *is completely positive if and only if there exist a Hilbert space* $\mathcal{K}$*, a bounded linear operator* $V : \mathcal{H} \to \mathcal{K}$ *and a* $*$*-homomorphism* $\pi : \mathbf{A} \to B(\mathcal{K})$ *such that*

$$\phi(a) = V^*\pi(a)V \quad \text{for all } a \in \mathbf{A}.$$

Furthermore $\|V\|^2 \leq \|\phi(1)\|$.

(ii) ϕ *is completely copositive if and only if there exist* $\mathcal{K}$ *and* V *as above and an anti-homomorphism* $\pi : \mathbf{A} \to B(\mathcal{K})$ *such that*

$$\phi(a) = V^*\pi(a)V \quad \text{for all } a \in \mathbf{A}.$$

Proof. The proof [Sti55] is analogous to the proof of Gelfand-Naimark-Segal construction . □

Similar to Stinespring's dilation Theorem, there is a finer structural result on completely positive maps and completely copositive maps between finite dimensional matrix algebras. We shall denote by $P_k[n,m]$ and $P^k[n,m]$ the set of all k-positive linear maps and the set of all k-copositive linear maps between $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$, respectively. $CP[n,m]$ and $CCP[n,m]$ are defined to be the set of all completely positive maps and the set of all completely copositive maps. Let $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ denote the linear maps from $M_n(\mathbb{C})$ to $M_n(\mathbb{C})$. The characterization of maps in $CP[n,m]$ is given by M. D. Choi in [Ch72]. First of all, we introduce the notion of the Choi matrix associated with a linear map.

Definition 1.1.5 *Let* ϕ *be a positive map in* $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$*. The Choi matrix of the map* ϕ *is defined by*

$$C_\phi = \sum_{i,j=1}^{n} E_{ij} \otimes \phi(E_{ij}),$$

where $\{E_{ij}\}_{i,j=1}^{n}$ *is the full set of canonical matrix units in* $M_n(\mathbb{C})$*,*

We shall denote by $n \wedge m$ and $n \vee m$ the minimum and maximum of n and m, resepectively. If $C = \sum_j A_j \otimes B_j \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$, define $C^\Gamma = \sum_j A_j^t \otimes B_j$.

Theorem 1.1.3 *Let* ϕ *be a positive map in* $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$*. The following are equivalent.*

(i) ϕ *is completely positive.*

(ii) C_ϕ *is positive semidefinite.*

(iii) ϕ *is* $m \wedge n$*-positive.*

Similarly, the following are equivalent.

(vi) ϕ *is completely copositive.*

(vii) C_ϕ^Γ *is positive semidefinite.*

(viii) ϕ *is* $m \wedge n$*-copositive.*

Here $C_\phi^\Gamma \triangleq \sum_{i,j=1}^n E_{ji} \otimes \phi(E_{ij})$.

As an immediate consequence, we have two chains of cones:

$$\begin{aligned} & P_1[n,m] \supseteq P_2[n,m] \supseteq \cdots \supseteq P_{n\wedge m}[m,n] = CP[m,n], \\ P_1[n,m] \quad &= P^1[n,m] \supseteq P^2[n,m] \supseteq \cdots \supseteq P^{n\wedge m}[n,m] = CCP[n,m]. \end{aligned}$$

The first example to distinguish $(k+1)$-positivity from k-positivity is introduced by Choi in [Ch72], hence the inclusions in the above two chains are strict. For this purpose, we include a more general result by Tomiyama in [To85, Theorem 2] here.

Example 1.1.1 *Let* $1 \leq k \leq n$ *and* $\phi_\lambda = \lambda tr_n + (1-\lambda) id_n$ $(-\infty < \lambda < \infty)$*, where* tr_n *is considered as the normalized trace operator defined by* $tr_n(x) = \frac{1}{n}\mathrm{Tr}_n(x) I_n$*, and* I_n *is the* $n \times n$ *identity matrix. Then* ϕ_λ *is* k*-positive if and only if* $0 \leq \lambda \leq 1 + \frac{1}{nk-1}$.

Let us denote by $D[n,m]$ the set of all decomposable maps in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$. We can implement this term into the chains:

$$P_1[n,m] = P^1[n,m] \supseteq D[n,m] \supseteq CP[n,m] \ (or\ CCP[n,m]).$$

It is an interesting question to determine the equality $P[n,m] = D[n,m]$. Or equivalently, do there exist indecomposable maps between matrix algebras? The following example of an indecomposable map from $M_3(\mathbb{C})$ to $M_3(\mathbb{C})$ given by Choi [Ch75] is indecomposable when $\mu \geq 1$. In particular, Φ_1 is usually called the Choi map.

Example 1.1.2 *The map $\Phi_\mu : M_3(\mathbb{C}) \to M_3(\mathbb{C})$ defined by*

$$\Phi_\mu(X) = \begin{bmatrix} x_{11} + \mu x_{33} & -x_{12} & -x_{13} \\ -x_{21} & \mu x_{11} + x_{22} & -x_{23} \\ -x_{13} & -x_{32} & \mu x_{22} + x_{33} \end{bmatrix} \textit{for} \quad X = [x_{ij}]_{i,j=1}^{3} \in M_3(\mathbb{C})$$

Later Woronowicz answered this question by giving an affirmative answer for the cases when $(n, m) = (2, 2), (2, 3), (3, 2)$ [Wo76] and gave an example of indecomposable map from $M_2(\mathbb{C})$ to $M_4(\mathbb{C})$, see also [HaKy16].

Theorem 1.1.4 *$P_1[n, m] = D[n, m]$ if and only if $nm \leq 6$.*

Proof. The proof by Woronowicz is based on long and ad hoc computations [Wo76]. □

Since Woronowicz published his paper in 1976, many endeavors to simplify the proof of Theorem 1.1.4 have been undertaken. The $(2, 2)$ case in Theorem 1.1.4 is usually called Størmer's theorem because Størmer obtained a general decomposition theorem in C^*-algebra setting which serves as the starting point for similar decomposition theorems [St63]. The most recent progresses towards reproving Størmer' theorem are a proof which adopts a geometrical view given by Miller and Olkiewicz [MiOl15] and another proof based on Brouwer's fixed point theorem given by Aubrun and Szarek [AuSz15]. Meanwhile, as far as I know, there is no alternative proof other than Woronowicz's original one for the $(2, 3)$ case in Theorem 1.1.4.

1.2 Quantum Entanglement

"Quantum entanglement is one but the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."

— Erwin Schödinger.

This note will focus on an important quantum measure, namely, Schmidt number, to shed light on the mathematical structure of quantum entanglement. We shall introduce some basic concepts in the mathematical formulation of quantum mechanics. Any given quantum system is identified with a Hilbert space $\mathcal{H}$. Since the Bra-ket notation is standard in quantum mechanics for describing quantum states, we will stick to this standard notation when introducing the following definitions of quantum states. However, we will use the mathematical notation in linear algebra when necessary. The dilemma lies in the fact that mathematicians and physicists both have their own ways of explaining quantum states in their own languages. We have to choose our standpoint in later sections to make this note simple and straight.

Notation 1.2.1 *Let $\mathcal{H}$ be an n-dimensional Hilbert space and we identify it with $\mathbb{C}^n$. The correspondence between the Bra-ket notations and the mathematical notations are the following.*

(a) $|\psi\rangle \in \mathcal{H}$ *corresponds to a column vector* $\xi \in \mathbb{C}^n$*;*

(b) $\langle\psi| \in B(\mathcal{H}, \mathbb{C})$ *corresponds to the complex conjugation* $\xi^* \in \mathbb{C}^n$*;*

(c) $\langle\phi|\psi\rangle$ *corresponds to the inner product* $\xi^*\eta$ *of the two vectors* $\eta \in \mathbb{C}^n$ *and* $\xi \in \mathbb{C}^n$*;*

(d) $|\phi\rangle\langle\psi| \in B(\mathcal{H})$ *corresponds to the operator* $\eta\xi^*$ *in* $M_n(\mathbb{C})$*.*

We are ready to formulate the mathematical definition of quantum states now.

Definition 1.2.2 *Let $\mathcal{H}$ be a Hilbert space associated with a quantum system. A state is a vector $|\psi\rangle$ in $\mathcal{H}$. Moreover, the state is normalized if $\langle\psi|\psi\rangle = 1$. The density matrix of the state is defined by $|\psi\rangle\langle\psi|$.*

In physicists' perspective, the word "state" refers to the vector or the corresponding density matrix in Definition 1.2.2. We shall use a small Greek letter to denote the density matrix of a state and identify it with the state for simplicity.

A bipartite quantum system is required so as to include the nature of quantum entanglement. Mathematically speaking, this composed quantum system is given by the tensor product of the two quantum subsystems. We shall briefly introduce the notion of tensor product for completeness. The following definitions are adapted from [St13].

Definition 1.2.3 *Let $\mathcal{K}$ and $\mathcal{H}$ be Hilbert spaces, and let $\{\xi_i\}_{i\in I}$, I an index set, an orthonormal basis (ONB) for $\mathcal{K}$. Let $\mathcal{H}_i = \mathcal{H}$ for $i \in I$, and let $\Pi = \oplus_{i\in I}\mathcal{H}_i$ be the Hilbert space direct sum of $\mathcal{H}_i$. For $\xi = \sum_{i\in I}\alpha_i\xi_i \in \mathcal{K}$ and $\eta \in \mathcal{H}$, define the product vector $\xi\otimes\eta \in \Pi$ by $\xi\otimes\eta = (\alpha_i\eta)_{i\in I}$ whose norm is inherited from $\mathcal{K}$ and $\mathcal{H}$ by $\|\xi\otimes\eta\|_\Pi \triangleq \sqrt{\sum_{i\in I}|\alpha_i|^2\|\eta\|_{\mathcal{H}}^2} = \|\xi\|_{\mathcal{K}}\|\eta\|_{\mathcal{H}}$.*

For example, if $\mathcal{K} = M_n(\mathbb{C})$ and $\mathcal{H} = M_m(\mathbb{C})$, we have $A\otimes B = [a_{ij}B]_{i,j=1}^n \in M_{nm,nm}(\mathbb{C})$ for $A = [a_{ij}]_{i,j=1}^n \in M_n(\mathbb{C})$ and $B = [b_{kl}]_{k,l=1}^m \in M_m(\mathbb{C})$, by Definition 1.2.3.

Definition 1.2.4 *Let $\mathcal{K}$ and $\mathcal{H}$ be Hilbert spaces. Using notations in Definition 1.2.3, the algebraic tensor product of $\mathcal{K}$ and $\mathcal{H}$ is defined as the linear span of the set of all product vectors. The tensor product of $\mathcal{K}$ and $\mathcal{H}$, written as $\mathcal{K} \otimes \mathcal{H}$, is defined as the completion of the algebraic tensor product of $\mathcal{K}$ and $\mathcal{H}$ in Π. An inner product on product vectors is inherited from inner products in $\mathcal{K}$ and $\mathcal{H}$ via the following:*

$$(\xi \otimes \eta, \psi \otimes \phi) \triangleq (\xi, \psi)(\eta, \phi), \quad \xi, \psi \in \mathcal{K}, \quad \eta, \phi \in \mathcal{H}.$$

We extend it bilinearly to make $\mathcal{K} \otimes \mathcal{H}$ a Hilbert space.

Note that Definition 1.2.4 is independent of the choice of orthonormal basis. With the definition of tensor product, we are ready to introduce the notion of entanglement.

Definition 1.2.5 *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a Hilbert space associated with two subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$, called a bipartite quantum system. Let $\{e_i^A \otimes e_j^B\}_{ij}$ be an ONB in $\mathcal{H}$, the Hilbert-Schmidt norm of ρ is defined as $\|\rho\|_{HS} = \sqrt{\sum_{ij} |a_{ij}|^2}$ if $\rho = \sum_{ij} a_{ij} e_i^A \otimes e_i^B$.*

(a) A pure state ρ is a vector $|\psi\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ with its density matrix defined by $|\psi\rangle\langle\psi|$.

(b) A mixed state ρ is given by a collection of pure states $\{p_i, |\psi_i\rangle\}$ with its density matrix defined by $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\sum_i p_i = 1$ and $p_i \geq 0\ \forall i$.

Further, the state ρ is said to be normalized if $\|\rho\|_{HS} = 1$, otherwise ρ is said to be unnormalized.

If $|\psi\rangle = |\psi_a\rangle \otimes |\psi_b\rangle$ is a product vector in $\mathcal{H}_A \otimes \mathcal{H}_B$, where $|\psi_a\rangle \in \mathcal{H}_A$ and $|\psi_b\rangle \in \mathcal{H}_B$, we can associate $|\psi\rangle$ with an operator $|\psi\rangle\langle\psi|$ in $B(\mathcal{H}_A \otimes \mathcal{H}_B)$ via $|\psi\rangle\langle\psi|(|\xi\rangle \otimes |\eta\rangle) \triangleq |\psi_a\rangle\langle\psi_a|\xi\rangle \otimes |\psi_b\rangle\langle\psi_b|\eta\rangle$, $\xi \in \mathcal{H}_A, \eta \in \mathcal{H}_B$ on product vectors and extend it linearly. On the other hand, the action of the tensor product of two operators $|\psi_a\rangle\langle\psi_a| \in B(\mathcal{H}_A)$ and $|\psi_b\rangle\langle\psi_b| \in B(\mathcal{H}_B)$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is $(|\psi_a\rangle\langle\psi_a| \otimes |\psi_b\rangle\langle\psi_b|)(\xi \otimes \eta) \triangleq |\psi_a\rangle\langle\psi_a|\xi\rangle \otimes |\psi_b\rangle\langle\psi_b|\eta\rangle$, $\xi \in \mathcal{H}_A, \eta \in \mathcal{H}_B$. Hence we obtain the fact that $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \cong B(\mathcal{H}_A \otimes \mathcal{H}_B)$.

So the density matrix of a quantum state ρ resides in $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$ and can be identified with a positive semidefinite matrix in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ if $\dim(\mathcal{H}_A) = n$ and $\dim(\mathcal{H}_B) = m$. Moreover, the equation $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ in Definition 1.2.5 (b) is a rank-1 decomposition of the state by definition. Assume that the dimensions of all Hilbert spaces involved are finite from now on. Two useful operations for quantum states are introduced.

Definition 1.2.6 *Let ρ be a state on a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. Denote by α and β two operators in $B(\mathcal{H}_A)$ and $B(\mathcal{H}_B)$, respectively.*

(a) The partial trace operators:

$\mathrm{Tr}_A : B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ *is defined by* $\mathrm{Tr}_A(\alpha \otimes \beta) = \mathrm{Tr}(\alpha)\beta$ *on product vectors and extends linearly to* $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$*.*

$\mathrm{Tr}_B : B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \to B(\mathcal{H}_A)$ *is defined by* $\mathrm{Tr}_B(\alpha \otimes \beta) = \mathrm{Tr}(\beta)\alpha$ *on product vectors and extends linearly to* $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$*.*

(b) The partial transpose operators:

$\Gamma_A : B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \to B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$ *is defined by* $\Gamma_A(\alpha \otimes \beta) = \alpha^t \otimes \beta$ *on product vectors and extends linearly to* $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$*.*

$\Gamma_B : B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \to B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$ *is defined by* $\Gamma_B(\alpha \otimes \beta) = \alpha \otimes \beta^t$ *on product vectors and extends linearly to* $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$*.*

We shall denote by $\rho_A = \mathrm{Tr}_B(\rho)$ and $\rho_B = \mathrm{Tr}_A(\rho)$ the *reduced states* of ρ w.r.t. the two subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. A state ρ is said to be a $k \times l$ bipartite state if $\mathrm{rank}(\rho_A) = k$ and $\mathrm{rank}(\rho_B) = l$. Moreover, a bipartite state ρ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is called positive partial transpose w.r.t. the subsystem A or B if $\Gamma_A(\rho) \geq 0$ or $\Gamma_B(\rho) \geq 0$, respectively. For simplicity, we shall use ρ^{Γ_A} and ρ^{Γ_B} to denote $\Gamma_A(\rho)$ and $\Gamma_B(\rho)$, respectively.

A pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is called a product state if $|\psi\rangle = |\psi_a\rangle \otimes |\psi_b\rangle$ holds for some $|\psi_a\rangle \in \mathcal{H}_A$ and $|\psi_b\rangle \in \mathcal{H}_B$, in which case we shall denote by $|\psi_a, \psi_b\rangle$ the product state $|\psi\rangle$. We review the definition of Schmidt number and its physical meanings [TeHo00].

Definition 1.2.7 *Let ρ be a state in a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. A decomposition $\{p_i \geq 0, |\psi_i\rangle\}$ of ρ is given by $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\sum_i p_i = 1$ and $p_i \geq 0$, $\forall i$.*

(a) Let $|\psi\rangle$ be a pure state. The Schmidt rank of $|\psi\rangle$, written as $\mathrm{sr}(|\psi\rangle)$*, is the minimum integer r such that there exist r product states $\{|\psi_{j,1}, \psi_{j,2}\rangle\}_{j=1,\dots,r}$ and $|\psi\rangle = \sum_{j=1}^{r} |\psi_{j,1}, \psi_{j,2}\rangle$. The Schmidt number* $\mathrm{SN}(\rho)$ *is defined to be the Schmidt rank* $\mathrm{sr}(\rho)$ *of the pure state.*

(b) Let ρ be a bipartite mixed state. The Schmidt number of ρ, written as $\mathrm{SN}(\rho)$*, is the integer k satisfying the following:*

(i) for any decomposition $\{p_i \geq 0, |\psi_i\rangle\}$ of ρ, at least one of the vectors $|\psi_i\rangle$ has Schmidt rank at least k, and

(ii) there exists a decomposition of ρ with all vectors $|\psi_i\rangle$ of Schmidt rank at most k.

The Schmidt rank of a pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ equals the rank of $|\psi\rangle = \sum_j |\psi_{j,1}\rangle \otimes |\psi_{j,2}\rangle$ as a matrix in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^n \otimes \mathbb{C}^m \cong M_{n,m}(\mathbb{C})$ if $\dim(\mathcal{H}_A) = n$ and $\dim(\mathcal{H}_B) = m$. Although the Schmidt number of pure states are easily attainable, the computation of Schmidt number for a mixed state can be a tedious task.

Physically, if two systems are independent from each other then measuring one of them should not affect the other. So the system can be described by the tensor product of two pos semidefinite matrices. Otherwise it is called "entangled" and the precise definition is given below.

Definition 1.2.8 *Let ρ be a state in the bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then ρ is said to be separable if it has Schmidt number one, otherwise ρ is said to be entangled.*

If ρ is separable, by definition it is the convex combination of product states, i.e., $\rho = \sum_i \alpha_i \otimes \beta_i$, where α_i and β_i are states on $\mathcal{H}_A$ and $\mathcal{H}_B$. The general separability problem has been proved to be NP-hard [Gu03, Gh10]. On the other hand, the only exceptions are the cases when $(\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)) = (2,2), (2,3), (3,2)$, where the well known Peres-Horodecki criterion tell us that all positive partial transpose states are separable [Pe96]. This criterion is equivalent to the statement of positive maps in Theorem 1.1.4 and it is not a coincidence.

Next we introduce two well known states to illustrate the notion of Schmidt numbers as well as entangled states. Let $\mathcal{H}$ be a d-dimensional Hilbert space and $\{|i\rangle\}_{i=1}^d$ an ONB of $\mathcal{H}$. We shall consider the states in the bipartite system $\mathcal{H} \otimes \mathcal{H}$. Denote by $|ij\rangle$ the product vector of $|i\rangle$ and $|j\rangle$.

Example 1.2.1 *The maximally entangled state is a pure state defined by $|\psi_0\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ in the bipartite system $\mathcal{H} \otimes \mathcal{H}$. By Definition 1.2.7, $\mathrm{SN}(|\psi_0\rangle) = \mathrm{sr}(|\psi_0\rangle) = d$.*

Denote by $\hat{U}$ the conjugate of a unitary operator U on $\mathcal{H}$.

Example 1.2.2 *An isotropic state ρ is a state in the bipartite system $\mathcal{H} \otimes \mathcal{H}$ which* *satisfies the following*

$$\rho = (U \otimes \hat{U})\rho(U \otimes \hat{U})^* \quad \textit{for all unitary operators } U \textit{ on } \mathcal{H}.$$

Further, every isotropic state has the form

$$\rho_F = F|\psi_0\rangle\langle\psi_0| + \frac{1-F}{d^2-1}(I_{d^2} - |\psi_0\rangle\langle\psi_0|). \quad F \in [0,1],$$

where $|\psi_0\rangle$ *is the maximally entangled state. It is shown in [HoHo99, GMS15] that*

(i) If $0 \le F \le \frac{1}{d}$, *then* ρ_F *is separable.*

(ii) If $\frac{k-1}{d} < F \le \frac{k}{d}$, *then* $\mathrm{SN}(\rho_F) = k$.

1.3 The Dual Cone Relation

Duality is a fundamental point of view in mathematics. The dual cone relation acts as a bridge between quantum states and positive linear maps. It was introduced and explained in a series of papers [Wo76, St86, It86, EoKy00] and soon became a necessity to investigate the two sides of the same coin.

Let X and Y be two real normed spaces and there is a bilinear form $\langle \cdot, \cdot \rangle$ defined on $X \times Y$. The duals of subsets $A \subseteq X$ and $B \subseteq Y$ are defined to be $A^\circ \triangleq \{y \in Y : \langle x, y\rangle \ge 0 \ \forall x \in A\}$ and $B^\circ \triangleq \{x \in X : \langle x, y\rangle \ge 0 \ \forall y \in B\}$ respectively. For $E \subseteq X$ and $F \subseteq Y$, the pairing (E, F) is called a dual pair w.r.t. the bilinear pairing if $E^\circ = F$ and $F^\circ = E$. Especially if (X, Y) is a dual pair, then we say that X and Y are dual to each other. If $E^\circ = F$ then we say that E is the pre-dual of F. The pre-dual of $B(\mathcal{H})$ is given by the space $T(\mathcal{H})$ of trace class operators via the duality $B(\mathcal{H}) \ni A \longmapsto \mathrm{Tr}(A\cdot)$. It suggests a duality between the space $B(\mathcal{A}, B(\mathcal{H}))$ of all bounded linear operators from a C^*-algebra $\mathcal{A}$ into $B(\mathcal{H})$ and the projective tensor product $\mathcal{A} \otimes_\pi T(\mathcal{H})$ as follows:

$$\langle x \otimes y, \phi\rangle = \mathrm{Tr}(\phi(x)y^t), \quad x \in \mathcal{A}, y \in T(\mathcal{H}), \phi \in B(\mathcal{A}, B(\mathcal{H})).$$

We shall employ the minimal amount of notations to introduce the dual cone relation in the matrix algebras setting. For this purpose, we identify $\mathcal{H}_A$ and $\mathcal{H}_B$ with $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Hence $B(\mathcal{H}_A)$ and $B(\mathcal{H}_B)$ is represented by $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$, respectively. A bipartite quantum state ρ is identified with a positive semidefinite matrix in $M_n(\mathbb{C})\otimes M_m(\mathbb{C})$. Denote by $V_k[n,m]$ and $V^k[n,m]$ the set $\{\rho\in M_n(\mathbb{C})\otimes M_m(\mathbb{C}) : \rho\geq 0,\ \mathrm{SN}(\rho)\leq k\}$ and the set $\{\rho\in M_n(\mathbb{C})\otimes M_m(\mathbb{C}) : \rho^{\Gamma_A}\geq 0,\ \mathrm{SN}(\rho^{\Gamma_A})\leq k\}$, respectively. We have two chains of cones:

$$V_1[n,m]\subseteq V_2[n,m]\subseteq\cdots\subseteq V_{n\wedge m}[m,n]=(M_n(\mathbb{C})\otimes M_m(\mathbb{C}))^+,$$
$$V_1[n,m]=V^1[n,m]\subseteq V^2[n,m]\subseteq\cdots\subseteq V^{n\wedge m}[n,m].$$

The aforementioned duality gives rise to the duality between the space $B(M_n(\mathbb{C}),M_m(\mathbb{C}))$ and the space $M_n(\mathbb{C})\otimes M_m(\mathbb{C})$ where the former contains the convex cone of all positive linear maps and the latter contains the convex cone of all quantum states. Denote by $\{e_{ij}\}_{i,j=1}^n$ the full set of matrix units in $M_n(\mathbb{C})$. For a map $\phi\in B(M_n(\mathbb{C}),M_m(\mathbb{C}))$ and a matrix $A=\sum_{i,j=1}^n e_{ij}\otimes a_{ij}$ in $M_n(\mathbb{C})\otimes M_m(\mathbb{C})$, the bilinear pairing is

$$\begin{aligned}
\langle A,\phi\rangle &= \sum_{i,j=1}^n\langle e_{ij}\otimes a_{ij},\phi\rangle\\
&=\mathrm{Tr}\left[\left(\sum_{i,j=1}^n e_{ij}\otimes\phi(e_{ij})\right)\left(\sum_{k,l=1}^n e_{lk}\otimes a_{kl}^t\right)\right]\\
&=\mathrm{Tr}\left(\sum_{i,j=1}^n\sum_{k,l=1}^n e_{ij}e_{lk}\otimes\phi(e_{ij}a_{kl}^t)\right)\\
&=\sum_{i,j=1}^n\mathrm{Tr}(\phi(e_{ij})a_{ij}^t)\\
&=\mathrm{Tr}(C_\phi A^t)=\mathrm{Tr}(AC_\phi^t),
\end{aligned}$$

where C_ϕ is the Choi matrix in Definition 1.1.5. For a vector ξ in $\mathbb{C}^n\otimes\mathbb{C}^m$, we have $\mathrm{sr}(\xi)=\dim(span\{z_i : \xi=\sum_i e_i\otimes z_i\})$. Theorem 1.3.1 is taken from [Ky13].

Theorem 1.3.1 *Let ϕ be a linear map in $B(M_n(\mathbb{C}),M_m(\mathbb{C}))$. we have the following:*

(1) The map ϕ is k-positive if and only if $\langle\xi\xi^*,\phi\rangle\geq 0,\ \forall\xi\in\mathbb{C}^n\otimes\mathbb{C}^m,\ \mathrm{sr}(\xi)\leq k$.

(2) The map ϕ is k-copositive if and only if $\langle(\xi\xi^*)^{\Gamma_A},\phi\rangle\geq 0,\ \forall\xi\in\mathbb{C}^n\otimes\mathbb{C}^m,\ \mathrm{sr}(\xi)\leq k$.

Proof. Refer to [Ky13] in the matrix algebra setting, also refer to [It86] in the C^*-algebra setting. □

Note that the $V_k[n,m]$ and $V^k[n,m]$ are convex hull of the set $\{\rho : \rho=\sum_i\xi\xi^*,\ \xi\in\mathbb{C}^n\otimes\mathbb{C}^m,\ \mathrm{sr}(\xi)\leq k\}$ and $\{\rho : \rho=\sum_i(\xi\xi^*)^{\Gamma_A},\ \xi\in\mathbb{C}^n\otimes\mathbb{C}^m,\ \mathrm{sr}(\xi)\leq k\}$, respectively. Hence by Theorem 1.3.1, one obtains $P_k[n,m]^\circ=\{\rho : \rho=\sum_i\xi\xi^*,\ \xi\in\mathbb{C}^n\otimes\mathbb{C}^m,\ \mathrm{sr}(\xi)\leq k\}^{\circ\circ}=V_k[n,m]$ and $P^k[n,m]^\circ=\{\rho : \rho=\sum_i(\xi\xi^*)^{\Gamma_A},\ \xi\in\mathbb{C}^n\otimes\mathbb{C}^m,\ \mathrm{sr}(\xi)\leq

$k\}^{\circ\circ} = V^k[n,m]$. Further, we obtain that $V_k[n,m]^\circ = P_k[n,m]^{\circ\circ} = P_k[n,m]$ and $V^k[n,m]^\circ = P^k[n,m]^{\circ\circ} = P^k[n,m]$ since $P_k[n,m]$ and $P^k[n,m]$ are convex sets. Hence $(V_k[n,m], P_k[n,m])$ and $(V^k[n,m], P^k[n,m])$ are dual pairs under the aforementioned bilinear pairing. This is called the dual cone relation between positive linear maps and quantum states which can be explained by the following two diagrams.

1. Towers containing the dual pairs $(P_k[n,m], V_k[n,m])$:

$$\begin{array}{ccccccccc} V_1[n,m] & \subsetneqq & V_2[n,m] & \subsetneqq & \cdots & \subsetneqq & V_{n\wedge m}[n,m] & = & (M_n(\mathbb{C})\otimes M_m(\mathbb{C}))^+ \\ \updownarrow & & \updownarrow & & & & \updownarrow & & \parallel \\ P_1[n,m] & \supsetneqq & P_2[n,m] & \supsetneqq & \cdots & \supsetneqq & P_{n\wedge m}[n,m] & \cong & (M_n(\mathbb{C})\otimes M_m(\mathbb{C}))^+ \end{array}.$$

2. Towers containing the dual pairs $(P^k[n,m], V^k[n,m])$:

$$\begin{array}{ccccccccc} V^1[n,m] & \subsetneqq & V^2[n,m] & \subsetneqq & \cdots & \subsetneqq & V^{n\wedge m}[n,m] & = & \{\rho : \rho^{\Gamma_A}\in (M_n(\mathbb{C})\otimes M_m(\mathbb{C}))^+\} \\ \updownarrow & & \updownarrow & & & & \updownarrow & & \parallel \\ P^1[n,m] & \supsetneqq & P^2[n,m] & \supsetneqq & \cdots & \supsetneqq & P^{n\wedge m}[n,m] & \cong & \{C : C^{\Gamma_A}\in (M_n(\mathbb{C})\otimes M_m(\mathbb{C}))^+\} \end{array}.$$

By the dual cone relation the Schmidt number of an entangled state $\rho \in M_n(\mathbb{C})\otimes M_m(\mathbb{C})$ can be rephrased in the following Lemma.

Lemma 1.3.2

$$\begin{aligned} \mathrm{SN}(\rho) \quad &= \max_l \{l : \exists\ \phi \in P_l[m,n]\ s.t.\ \langle \rho, \phi\rangle < 0\} + 1, &(1.1)\\ &= \min_l \{l : \langle \rho, \phi\rangle \geq 0\ \forall\ \phi \in P_l[m,n]\}. &(1.2)\end{aligned}$$

If $\langle \rho, \phi\rangle < 0$, then ϕ is called an entanglement witness by which one can detect that ρ is entangled [Te00]. If such a map exists, then the detected state has Schmidt number at least two. To decide the Schmidt number of ρ, one should continue to test ρ using k-positive maps as entanglement witnesses up to certain k, where no k-positive map can serve as an entanglement witness to ρ. Recently, Aubrun and J. Szarek showed that the number of such maps to detect all the robustly entangled states (i.e. $\rho \in M_d(\mathbb{C})\otimes M_d(\mathbb{C})$ is robustly entangled if $\frac{1}{2}(\rho + I_{d^2}/2))$ exceeds $\exp(Cd^3/\log d)$ [AuSz16].

Note that for two closed convex cones C and D in X, one have $(C+D)^\circ = C^\circ \cap D^\circ$ and $(C\cap D)^\circ = C^\circ + D^\circ$. Hence $(V_s[n,m]\cap V^t[n,m], P_s[n,m]+P^t[n,m])$ is also a dual pair. The dual object of decomposable maps $D[n,m]$, denoted by $T[n,m]$, equals $V_{n\wedge n}[n,m]\cap V^{n\wedge m}[n,m]$. We can implement the dual pair $(T[n,m], D[n,m])$ into the above diagram like this:

$$\begin{array}{ccccccc}
V_1[n,m] & \subseteq & T[n,m] & \subseteq & V_{n\wedge m}[n,m] & = & (M_n(\mathbb{C})\otimes M_m(\mathbb{C}))^+ \\
\updownarrow & & \updownarrow & & \updownarrow & & \| \\
P_1[n,m] & \supseteq & D[n,m] & \supseteq & P_{n\wedge m}[n,m] & \cong & (M_n(\mathbb{C})\otimes M_m(\mathbb{C}))^+
\end{array}.$$

In low dimensional cases ($(3,2)$ is omitted by symmetry), we list the diagrams by Theorem 1.1.4 and Example 1.1.1:

$$\begin{array}{ccccccc}
V_1[2,2] & = & T[2,2] & \subsetneqq & V_2[2,2] & = & (M_2(\mathbb{C})\otimes M_2(\mathbb{C}))^+ \\
\updownarrow & & \updownarrow & & \updownarrow & & \| \\
P_1[2,2] & = & D[2,2] & \supsetneqq & P_2[2,2] & \cong & (M_2(\mathbb{C})\otimes M_2(\mathbb{C}))^+
\end{array},$$

$$\begin{array}{ccccccc}
V_1[2,3] & = & T[2,3] & \subsetneqq & V_2[2,3] & = & (M_2(\mathbb{C})\otimes M_3(\mathbb{C}))^+ \\
\updownarrow & & \updownarrow & & \updownarrow & & \| \\
P_1[2,3] & = & D[2,3] & \supsetneqq & P_2[2,3] & \cong & (M_2(\mathbb{C})\otimes M_3(\mathbb{C}))^+
\end{array}.$$

The Peres-Horodecki criterion [Pe96] is shown by the equalities in the first column of the above two diagrams. It says that a 2×2 or 2×3 quantum state ρ is separable if and only if ρ is PPT, namely $\rho^{\Gamma_A}\geq 0$.

When $(n,m)=(2,4)$, recall the example given by Woronowicz in [Wo76], then

$$\begin{array}{ccccccc}
V_1[2,4] & \supsetneqq & T[2,4] & \subsetneqq & V_2[2,4] & = & (M_2(\mathbb{C})\otimes M_4(\mathbb{C}))^+ \\
\updownarrow & & \updownarrow & & \updownarrow & & \| \\
P_1[2,4] & \subsetneqq & D[2,4] & \supsetneqq & P_2[2,4] & \cong & (M_2(\mathbb{C})\otimes M_4(\mathbb{C}))^+
\end{array}.$$

In general, we cannot implement the dual pair $(T[n,m], D[n,m])$ into the diagram since we do not know the inclusion relation between $T[n,m]$ and $V_k[n,m]$ when $1<k<n\wedge m$.

For the diagram in the case $(n,m)=(3,3)$, it was conjectured that $T[3,3] \subseteq V_2[3,3]$ in [SBL01]. This is equivalent to the claim that $P_2[3,3] \subseteq D[3,3]$, namely, every 2-positive linear map from $M_3(\mathbb{C})$ to $M_3(\mathbb{C})$ is decomposable.

The elements in $T[n,m] \backslash V_1[n,m] = \{\rho : \rho \geq 0, \rho^{\Gamma_A} \geq 0, \mathrm{SN}(\rho) \geq 2\}$ are called positive partial transpose entangled states (PPTES). The set $T[n,m]$ takes a constant portion in the set $V_{n\wedge m}[n,m]$ under the α-volume [Ye10]. Further, the authors in [ASY13] showed most random reduced states are entangled. It is interesting to investigate the α-volume of $V_k[n,m]$.

1.4 Main Results

In Chapter 2, we show a decomposition theorem for k-positive maps.

Theorem 2.2.2 *Let ϕ be a non-zero k-positive $(2 \leq k < n \wedge m)$ map in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$. Then there exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely positive map and γ is a p-trivial lifting of a $(k-1)$-positive map in $B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$, for some $p \in \{1, ..., n\}$.*

As a consequence, we give an affirmative answer to the conjecture that every 3×3 bipartite PPTES has Schmidt number 2 [SBL01]. Hence we can complete the diagram when $(n,m) = (3,3)$ as below.

$$\begin{array}{ccccccccc} V_1[3,3] & \subsetneqq & T[3,3] & \subsetneqq & V_2[3,3] & \subsetneqq & V_3[3,3] & = & (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \| \\ P_1[3,3] & \supsetneqq & D[3,3] & \supsetneqq & P_2[3,3] & \supsetneqq & P_3[3,3] & \cong & (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \end{array} .$$

In Chapter 3, we provide a systematical way to construct PPTES of any prescribed Schmidt number. Especially, we have the following result.

Theorem 3.2.17 *Given any positive integer r, there exist positive integers M, N and a bipartite PPT entangled state $\rho \in M_M(\mathbb{C}) \otimes M_N(\mathbb{C})$ of Schmidt number r.*

Equivalently, one may rephrase Theorem 3.2.17 into the language of positive maps.

Theorem 3.2.18 *Give any positive integer r, there exist positive integers M, N and* *an indecomposable map* $\phi \in B(M_M(\mathbb{C}), M_N(\mathbb{C}))$ *which is* r*-positive but not* $(r+1)$*-positive.*

Chapter 2

A Decomposition Theorem

Part of this chapter is adopted from our paper [YDT16] which focuses on k-positive linear maps between matrix algebras.

2.1 Background and Current Status

Since the complete convex structure of $P_1[n,m]$ is unknown when $nm \geq 8$, dedicated examples are introduced to analyze certain extremal points of $P_1[n,m]$. The problem we shall consider originates from the positive maps constructed by Cho, Kye and Lee in [CKL92]. It is a generalization of Choi's first example of indecomposable map in [Ch75].

Example 2.1.1 *The generalized Choi (GC) map $\Phi[a,b,c]$ is defined by*

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11}+ax_{22}+bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11}+cx_{22}+ax_{33} \end{pmatrix}$$

for $X = [x_{ij}] \in M_3(\mathbb{C})$, where a,b,c are nonnegative and real numbers. Let us use the convention that $P \wedge Q$ means conditions P and Q are both satisfied. $P \vee Q$ means either condition P or condition Q is satisfied, and $P \to Q$ means condition P implies condition Q. The map $\Phi[a,b,c]$ is

(i) positive if and only if $[a+b+c \geq 2] \wedge [0 \leq a \leq 1 \to bc \geq (1-a)^2]$,

(ii) 2-positive if and only if $[a \geq 2] \vee [[1 \leq a < 2] \wedge [bc \geq (2-a)(b+c)]]$,

(iii) completely positive if and only if $a \geq 2$,

(iv) 2-copositive if and only if it is completely co-positive if and only if $bc \geq 1$*,*

(v) decomposable if and only if $0 \leq a \leq 2 \to bc \geq (\frac{2-a}{2})^2$. □

It is obvious that 2-positivity (resp. 2-copositivity) implies decomposability for GC maps. Further, it was proved in [Ha98] the map $\Phi[a,b,c]$ is indecomposable if and only if it is atomic.

Later in [SBL01], Sanpera, Bruß, and Lewenstein presented evidence for special cases and posed the following conjecture:

Conjecture 2.1.1 *All PPTES in* $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ *have Schmidt number 2.*

They introduced the notion of k-edge state as follows:

Definition 2.1.1 *A state* $\delta \in V_{n \wedge m}[n,m]$ *is said to be a* k*-edge state if* $\delta - \epsilon \xi_k \xi_k^*$ *is not positive, for any* $\epsilon > 0$ *and Schmidt rank* k *vector* ξ_k*.*

Denote by $K(\rho)$, $R(\rho)$ and $r(\rho)$ the kernel, range, and rank of a state ρ. It is obvious that a mixed state δ is a k-edge state if and only if there exist no Schmidt rank k vector ξ_k such that $\xi_k \in R(\delta)$. Given any state ρ in $V_k[n,m]$, there exists a canonical decomposition of ρ which associates ρ with a k-edge state.

Lemma 2.1.1 *A state* $\rho_k \in V_k[n,m]$ *can be written as a convex combination of a state* ρ_{k-1} *in* $V_{k-1}[n,m]$ *and a* k*-edge state* δ*:*

$$\rho_k = (1-p)\rho_{k-1} + p\delta, \qquad 1 \geq p > 0,$$

where the edge state δ *has Schmidt number* $\geq k$*.*

According to Lemma 2.1.1, it suffices to prove Conjecture 2.1.1 for all the edge states. The birank of a PPTES ρ is $(\operatorname{rank}\rho, \operatorname{rank}\rho^{\Gamma_A})$. For example, the maximal birank of a $3 \otimes 3$ PPTES is $(9,9)$. As a partial result, the following has been shown in [SBL01, Lemma 3].

Theorem 2.1.2 *All PPTES in* $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ *of birank* $(4,4)$ *have Schmidt number 2.*

As mentioned in [SBL01], using a similar argument, one obtains that all PPTES in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ of birank $(n+1, m+1)$ have Schmidt number $\leq (n \wedge m) - 1$. Unfortunately, this methodology does not apply for the remaining types of edge states.

The full classification of edge states in $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ is far from complete, and one may refer to [KCKL00, HLVC00, KKL11, Ky13] for some partial results. The possible types of edge states in $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ are those states with birank $(r(\rho), r(\rho^{\Gamma_A}))$ given by:

$$(4,4), \quad (5,5) \quad (5,6) \quad (5,7) \quad (6,6) \quad (5,8) \quad (6,7) \quad (6,8),$$

under the condition (s,t), $s \leq t$ by symmetry. Concrete examples of various types of edge states are given in [Cl06, Ha07, KyOs12], but only for $(4,4)$ type edge states there is a structural result by Chen and Djoković [ChDj11], and Showronek [Sk11] independently that all PPT states in $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ of birank $(4,4)$ arise essentially from unextendible product bases.

Our approach towards Conjecture 2.1.1 is to peel off a completely positive map from a 2-positive map. That is, find a completely positive map which is dominated by the 2-positive map. Moreover, the dimension of the space where the remaining map resides is reduced. Indeed, this is a dimension-lowering trick. The technique of peeling off a completely positive map from a 2-positive map originates in [Ma10, Theorem 3.3]. Recall that a positive linear map ϕ is extremal if any other positive linear map ψ satisfying $\psi \leq \phi$ is either a positive multiple of ϕ or zero.

Theorem 2.1.3 *Let ϕ be a non-zero extremal map in $B^+(B(\mathcal{K}), B(\mathcal{H}))$. If ϕ is 2-positive (resp. 2-copositive), then it is completely positive (resp. completely copositive).*

In the proof of Theorem 2.1.3, Marciniak actually constructed a non-zero completely positive (resp. completely copositive) map which is dominated by the given non-zero 2-positive map (resp. 2-copositive). Let us review his construction by starting with a bounded positive linear mapping ϕ from $B(\mathcal{K})$ into $B(\mathcal{H})$, where $\mathcal{K}$ and $\mathcal{H}$ can be infinite dimensional. If ϕ is non-zero then there exist unit vectors $\xi \in \mathcal{K}$, $x \in \mathcal{H}$ and a positive number λ such that

$$\diamondsuit \quad \phi(\xi\xi^*)x = \lambda x.$$

For such fixed ξ, x and λ, define two bounded linear operators $B, C : \mathcal{K} \to \mathcal{H}$ by

$$\begin{aligned} B\eta &= \lambda^{-1/2}\phi(\eta\xi^*)x, \\ C\eta &= \lambda^{-1/2}\phi(\xi\overline{\eta}^*)x, \end{aligned}$$

where $\eta \in \mathcal{K}$ and $\overline{\eta}$ denotes the complex conjugate of η. Consequently, define two maps ψ_1 and ψ_2 from $B(\mathcal{K})$ into $B(\mathcal{H})$ by

$$\psi_1(X) = BXB^*,$$
$$\psi_2(X) = CX^tC^*,$$

for $X \in B(\mathcal{K})$. Using the above notations, Marciniak showed the following result.

Theorem 2.1.4 *Let $\phi : B(\mathcal{K}) \to B(\mathcal{H})$ be a positive non-zero map. Let ξ, x and λ fulfill $\diamondsuit$ and the operator B (resp. C) and the map $\psi_1(X) = BXB^*$ (resp. $\psi_2(X) = CX^tC^*$) be defined above. Then $\psi_1 \leq \phi$ if and only if for any $\eta \in \mathcal{K}$ and $y \in \mathcal{H}$ the following inequality holds*

$$|\langle y, \phi(\eta\xi^*)\rangle|^2 \leq \langle x, \phi(\xi\xi^*)x\rangle\langle y, \phi(\eta\eta^*)y\rangle.$$

Analogously, $\psi_2 \leq \phi$ if and only if for any $\eta \in \mathcal{K}$ and $y \in \mathcal{H}$ the following inequality holds

$$|\langle y, \phi(\xi\eta^*)\rangle|^2 \leq \langle x, \phi(\xi\xi^*)x\rangle\langle y, \phi(\eta\eta^*)y\rangle.$$

The key observation is that the condition $|\langle y, \phi(\eta\xi^*)\rangle|^2 \leq \langle x, \phi(\xi\xi^*)x\rangle\langle y, \phi(\eta\eta^*)y\rangle$ is linked to 2-positivity of the map ϕ. Similarly, one can obtain the other condition $|\langle y, \phi(\xi\eta^*)\rangle|^2 \leq \langle x, \phi(\xi\xi^*)x\rangle\langle y, \phi(\eta\eta^*)y\rangle$ from 2-copositivity of the map ϕ. For detailed proofs and discussions, see also Størmer's book [St13, pages 38-39]. Later Størmer obtained in [St13, Theorem 5] a decomposition for positive maps .

Definition 2.1.2 *A decomposable map $\alpha : \mathcal{A} \to B(\mathcal{H})$, $\alpha \leq \phi$ is a maximal decomposable map majorized by ϕ if there is no decomposable map $\psi : \mathcal{A} \to B(\mathcal{H})$ such that $\psi \neq \alpha$, and $\alpha \leq \psi \leq \phi$.*

Definition 2.1.3 *A positive map $\phi : \mathcal{A} \to B(\mathcal{H})$ is said to be optimal (resp. co-optimal) if $\psi \leq \phi$ for ψ completely positive (reps. co-positive) implies $\psi = 0$. Moreover, ϕ is said to be bi-optimal if it is both optimal and co-optimal.*

Theorem 2.1.5 *Let $\mathcal{A}$ be a finite dimensional C^*-algebra and $\mathcal{H}$ a finite dimensional Hilbert space. Let $\phi : \mathcal{A} \to B(\mathcal{H})$ be a positive linear map. Then there are a maximal decomposable map $\alpha : \mathcal{A} \to B(\mathcal{H})$ majorized by ϕ and a bi-optimal, hence atomic, map $\beta : \mathcal{A} \to B(\mathcal{H})$ such that $\phi = \alpha + \beta$.*

It is a structural result involving the use of Zorn's Lemma. It shows no clue of how to obtain such a decomposition for a general positive linear map from $\mathcal{A}$ into $B(\mathcal{H})$, and there is no effective decomposition algorithm even for positive linear maps in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$.

Theorem 2.1.4 was shown by M.-D. Choi by a block-matrix approach. Inspired by his insight, we present a stronger version of Theorem 2.1.4 in the following section.

2.2 A Decomposition Theorem

We shall give a useful definition below. Throughout this section, assume $2 \leq k \leq n \wedge m$.

Definition 2.2.1 (Trivial Lifting) *Given a linear map $\chi \in B(M_l(\mathbb{C}), M_m(\mathbb{C}))$, fix the canonical matrix unit basis E_{ij} $i, j = 1, .., l$, in $M_l(\mathbb{C})$, under which the Choi matrix is $C_\chi = [\chi(E_{ij})]_{i,j=1}^l \in M_l(M_m(\mathbb{C}))$. Given $I = \{n_1, ..., n_q\} \subset \{1, ..., l+q\}$, where $n_1 < \cdots < n_q$, extend the matrix C_χ to a $(l+q) \times (l+q)$ block matrix $C_I^{lift} \in M_{l+q}(M_m(\mathbb{C}))$ by adding one row and one column of $m \times m$ zero matrices at the n_k^{th} level for each $k = 1, ..., q$ as follows:*

$$C_I^{lift} \triangleq \begin{array}{c} \\ 1^{st} \\ \vdots \\ n_k^{th} \\ \vdots \\ (l+q)^{th} \end{array} \begin{array}{c} \begin{array}{ccccc} 1^{st} & \cdots & n_k^{th} & \cdots & (l+q)^{th} \end{array} \\ \begin{pmatrix} \chi(E_{11}) & \cdots & 0 & \cdots & \chi(E_{1,l}) \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \chi(E_{l,1}) & \cdots & 0 & \cdots & \chi(E_{l,l}) \end{pmatrix} \end{array}.$$

Denote by $\tilde{\chi}_I$ the map in $B(M_{l+q}(\mathbb{C}), M_m(\mathbb{C}))$ associated with the Choi matrix $C_{\tilde{\chi}_I} = [\tilde{\chi}_I(E_{st})]_{s,t=1}^{l+q} = C_I^{lift}$. Then the map $\tilde{\chi}_I$ is called a (I-)trivial lifting of the original map χ. If $I = \{p\}$ is a singleton, simply denote by $\tilde{\chi}_p$ the p-trivial lifting of χ.

For simplicity, we consider a positive linear map χ in $B(M_{n-1}(\mathcal{C}), M_m(\mathcal{C}))$ in the following Lemma.

Lemma 2.2.1 *The map χ is k-positive or k-copositive if and only if the p-trivial lifting $\tilde{\chi}_p$ is k-positive or k-copositive, respectively.*

Proof. Let $\eta = (w^1, ..., w^k)^t$ be an arbitrary column vector in $\mathbb{C}^k \otimes \mathbb{C}^n$ where $w^s \in \mathbb{C}^n$, $s = 1, ..., k$. Let $\hat{w}^s \in \mathbb{C}^{n-1}$ be defined as $(w_1^s, ..., w_{p-1}^s, w_{p+1}^s, ..., w_m^s)^t$ for

$s = 1, ..., k$, and $\hat{\eta} = (\hat{w}^1, ..., \hat{w}^k) \in \mathbb{C}^k \otimes \mathbb{C}^{n-1}$. By definition of p-trivial lifting,

$$(id_k \otimes \tilde{\chi}_p)(\eta\eta^*) = [\tilde{\chi}_p(w^s(w^t)^*)]_{s,t=1}^k = [\chi(\hat{w}^s(\hat{w}^t)^*)]_{s,t=1}^k = (id_k \otimes \chi)(\hat{\eta}\hat{\eta}^*).$$

This matrix equality in $M_k(M_m(\mathbb{C}))$ shows that the pair of maps $(\chi, \tilde{\chi})$ are k-positive simultaneously. For k-copositivity, we also have:

$$(\tau_k \otimes \tilde{\chi}_p)(\eta\eta^*) = [\tilde{\chi}_p(w^t(w^s)^*)]_{s,t=1}^k = [\chi(\hat{w}^t(\hat{w}^s)^*)]_{s,t=1}^k = (\tau_k \otimes \tilde{\chi})(\hat{\eta}\hat{\eta}^*).$$

This completes the proof. $\square$

By repeatedly using Lemma 2.2.1, a map χ is k-positive or k-copositive, if and only if its trivial lifting $\tilde{\chi}_I$ is k-positive or k-copositive, respectively.

Let us consider k-positive maps for the moment. A similar theorem holds for k-copositive maps.

Theorem 2.2.2 (Choi Decomposition) *Let $\phi \in B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ be a non-zero k-positive map. Then there exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely positive map and γ is a p-trivial lifting of a $(k-1)$-positive map in $B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$, for some $p \in \{1, ..., n\}$.*

Before proving Theorem 2.2.2, recall a classical result from [Bh07, Exercise 1.3.5]:

Lemma 2.2.3 *Suppose a hermitian matrix M is partitioned as*

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where A and C are square matrices. Then the following are equivalent:

(i) $M \geq 0$,

(ii) $A \geq 0,\ M/A \triangleq C - B^*A^\dagger B \geq 0,\ range(B) \subset range(A)$,

(iii) $C \geq 0,\ M/C \triangleq A - BC^\dagger B^* \geq 0,\ range(B^*) \subset range(C)$.

Here $A^\dagger$ and $C^\dagger$ refer to the Moore-Penrose pseudo inverses of A and C, respectively.

Some properties [BeGr03, Page 29-30] of the Moore-Penrose pseudo inverse $A^\dagger$ of a matrix A are useful.

P1 $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$.

P2 $(AA^\dagger)^* = AA^\dagger, \ (A^\dagger A)^* = A^\dagger A$.

P3 $AA^\dagger$ is the orthogonal projection onto the range of A, $A^\dagger A$ is the orthogonal projection onto the range of A^*.

P4 If A is invertible, then $A^\dagger = A^{-1}$.

P5 If $A \geq 0$, then $A^\dagger \geq 0$.

Proof. [**Theorem 2.2.2**] Since the k-positive map ϕ is non-zero, with respect to the canonical matrix units $\{E_{ij}\}_{i,j=1}^n$ in $M_n(\mathbb{C})$, there exists an index $k \in \{1, 2, ..., n\}$ such that $\phi(E_{kk}) \neq 0$. Otherwise if $\phi(E_{kk}) = 0$ for every $k = 1, .., n$, then $\phi(I_n) = 0$. Meanwhile for every $A \in M_n(\mathbb{C})^+$, $||A||I_n - A \geq 0$ yields that $0 = ||A||\phi(I_n) \geq \phi(A)$, implying $\phi = 0$, which contradicts $\phi \neq 0$. Without loss of generality, we assume that $\phi(E_{nn}) \neq 0$. Decompose the Choi matrix C_ϕ for ϕ, with $A_{ij} = \phi(E_{ij}), \ i, j = 1, ..., n$, as follows:

$$
\begin{aligned}
C_\phi &= \begin{pmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nj} & \cdots & A_{nn} \end{pmatrix} \\
&= \begin{pmatrix} A_{1n}A_{nn}^\dagger A_{n1} & \cdots & A_{1n}A_{nn}^\dagger A_{nj} & \cdots & A_{1n}A_{nn}^\dagger A_{nn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{in}A_{nn}^\dagger A_{n1} & \cdots & A_{in}A_{nn}^\dagger A_{nj} & \cdots & A_{in}A_{nn}^\dagger A_{nn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{nn}A_{nn}^\dagger A_{n1} & \ddots & A_{nn}A_{nn}^\dagger A_{nj} & \cdots & A_{nn}A_{nn}^\dagger A_{nn} \end{pmatrix} \\
&+ \begin{pmatrix} A_{11} - A_{1n}A_{nn}^\dagger A_{n1} & \cdots & A_{1j} - A_{1n}A_{nn}^\dagger A_{nj} & \cdots & A_{1n} - A_{1n}A_{nn}^\dagger A_{nn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} - A_{in}A_{nn}^\dagger A_{n1} & \cdots & A_{ij} - A_{in}A_{nn}^\dagger A_{nj} & \cdots & A_{in} - A_{in}A_{nn}^\dagger A_{nn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n1} - A_{nn}A_{nn}^\dagger A_{n1} & \ddots & A_{nj} - A_{nn}A_{nn}^\dagger A_{nj} & \cdots & A_{nn} - A_{nn}A_{nn}^\dagger A_{nn} \end{pmatrix} \\
&\triangleq U + R = C_\psi + C_\gamma.
\end{aligned}
$$

The two linear maps ψ and γ above are defined by the matrices U and R, respectively. For $i, j = 1, ..., n$, the (i, j)-entry of the matrix U is given by $A_{in}A_{nn}^\dagger A_{nj}$, and the (i, j)-entry of the matrix R is given by $R_{ij} = A_{ij} - A_{in}A_{nn}^\dagger A_{nj}$. Since $U \geq 0 \iff$

$\langle Ux, x\rangle \geq 0 \; \forall x \in \mathbb{C}^{nm}$, we deduce that

$$U = \begin{pmatrix} A_{1n} \\ \vdots \\ A_{in} \\ \vdots \\ A_{nn} \end{pmatrix} A_{nn}^{\dagger} \begin{pmatrix} A_{n1} & \cdots & A_{nj} & \cdots & A_{nn} \end{pmatrix} \geq 0.$$

$U \neq 0$ because its (n,n)-entry is $A_{nn}A_{nn}^{\dagger}A_{nn} = A_{nn} = \phi(E_{nn}) \neq 0$. Hence the map $\psi \neq 0$ corresponding to the matrix U is completely positive. By k-positivity of ϕ, for arbitrary column vectors $w^1, w^2, .., w^{k-1} \in \mathbb{C}^n$, taking $\xi = (w^1, ..., w^{k-1}, e_n)^t$ where $e_n = (0, ..., 0, 1)^t \in \mathbb{C}^n$, we have

$$\xi\xi^* = \begin{pmatrix} w^1(w^1)^* & \cdots & w^1(w^j)^* & \cdots & w^1 e_n^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ w^i(w^1)^* & \cdots & w^i(w^j)^* & \cdots & w^i e_n^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ e_n(w^1)^* & \cdots & e_n(w^j)^* & \cdots & e_n e_n^* \end{pmatrix} \geq 0$$

$$\Longrightarrow (id_k \otimes \phi)(\xi\xi^*) = \begin{pmatrix} \phi(w^1(w^1)^*) & \cdots & \phi(w^1(w^j)^*) & \cdots & \phi(w^1 e_n^*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi(w^i(w^1)^*) & \cdots & \phi(w^i(w^j)^*) & \cdots & \phi(w^i e_n^*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi(e_n(w^1)^*) & \cdots & \phi(e_n(w^j)^*) & \cdots & \phi(e_n e_n^*) \end{pmatrix} \geq 0.$$

By Lemma 2.2.3 (iii), the condition $(id_k \otimes \phi)(\xi\xi^*) \geq 0$ expands to:

$$\begin{pmatrix} \phi(w^1(w^1)^*) & \cdots & \phi(w^1(w^{k-1})^*) \\ \vdots & \ddots & \vdots \\ \phi(w^{k-1}(w^1)^*) & \cdots & \phi(w^{k-1}(w^{k-1})^*) \end{pmatrix} \geq \begin{pmatrix} \phi(w^1 e_n^*) \\ \vdots \\ \phi(w^{k-1} e_n^*) \end{pmatrix} \phi(e_n e_n^*)^\dagger \begin{pmatrix} \phi(e_n(w^1)^*) & \cdots & \phi(e_n(w^{k-1})^*) \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} \phi(w^1(w^1)^*) - \phi(w^1 e_n^*)\phi(e_m e_m^*)^\dagger \phi(e_n(w^1)^*) & \cdots & \phi(w^1(w^{k-1})^*) - \phi(w^1 e_n^*)\phi(e_n e_n^*)^\dagger \phi(e_n(w^{k-1})^*) \\ \vdots & \ddots & \vdots \\ \phi(w^{k-1}(w^1)^*) - \phi(w^{k-1} e_n^*)\phi(e_n e_n^*)^\dagger \phi(e_n(w^1)^*) & \cdots & \phi(w^{k-1}(w^{k-1})^*) - \phi(w^{k-1} e_n^*)\phi(e_n e_n^*)^\dagger \phi(e_n(w^{k-1})^*) \end{pmatrix} \geq 0.$$

For the (s,t) entry in the above matrix, by linearity,

$$\begin{aligned}
&\phi(w^s e_n^*)\phi(e_n e_n^*)^\dagger \phi(e_n(w^t)^*) \\
&= \phi\Big(\sum_{i=1}^n w_i^s e_i e_n^*\Big)\phi(e_n e_n^*)^\dagger \phi\Big(\sum_{j=1}^n \overline{w_j^t} e_n e_j^*\Big) \\
&= \Big(\sum_{i=1}^n w_i^s \phi(E_{in})\Big)\phi(E_{nn})^\dagger \Big(\sum_{j=1}^n \overline{w_j^t}\phi(E_{nj})\Big) \\
&= \sum_{i=1}^n \sum_{j=1}^n w_i^s \overline{w_j^t}\Big(\phi(E_{in})\phi(E_{nn})^\dagger \phi(E_{nj})\Big) \\
&= \sum_{i=1}^n \sum_{j=1}^n w_i^s \overline{w_j^t}(A_{in}A_{nn}^\dagger A_{nj}) \\
&= \sum_{i=1}^n \sum_{j=1}^n w_i^s \overline{w_j^t} U_{ij} \\
&= \sum_{i=1}^n \sum_{j=1}^n w_i^s \overline{w_j^t} \psi(e_i e_j^*) \\
&= \psi(w^s(w^t)^*).
\end{aligned}$$

Let $\gamma \triangleq \phi - \psi$, one has

$$\begin{pmatrix} \gamma(w^1(w^1)^*) & \cdots & \gamma(w^1(w^{k-1})^*) \\ \vdots & \ddots & \vdots \\ \gamma(w^{k-1}(w^1)^*) & \cdots & \gamma(w^{k-1}(w^{k-1})^*) \end{pmatrix} \geq 0, \ \forall w^1, ..., w^{k-1} \in \mathbb{C}^n,$$

proving that γ is $(k-1)$-positive. Moreover, all the entries of the n^{th} row and n^{th} column of the matrix R are zero matrices. To show this, recall ϕ is 2-positive($k \geq 2$), hence any sub-block $\begin{pmatrix} \phi(E_{nn}) & \phi(E_{nj}) \\ \phi(E_{jn}) & \phi(E_{jj}) \end{pmatrix} \geq 0$, for all $j = 1, ..., n-1$. By Lemma 2.2.3, one obtains that $ColumnSpace(\phi(E_{nj})) \subseteq ColumnSpace(\phi(E_{nn}))$, for all $j = 1, .., n$. By (P3), $A_{nn}A_{nn}^{\dagger}$ is the orthogonal projection onto the column space of A_{nn}, so $R_{nj} = A_{nj} - A_{nn}A_{nn}^{\dagger}A_{nj} = 0$, for all $j = 1, \ldots, n$. Note that R is hermitian, hence $R_{jn} = 0$ for all $j = 1, \ldots, n$. Denote the matrix $R = C_{\gamma}$ by:

$$R = \begin{pmatrix} & & 0 \\ K & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} & & 0 \\ C_{\kappa} & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Here, the map $\kappa \in B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$ is defined by the Choi matrix $K \in M_{(n-1)}(M_m(\mathbb{C}))$ through $\kappa(E_{st}) = K_{st}, s, t = 1, .., n-1$. It is obvious that $\gamma \in B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ is the n-trivial lifting of $\kappa \in B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$. By Lemma 2.2.1, the map $\kappa \in B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$ is $(k-1)$-positive. $\square$

A similar result holds for k-copositive maps.

Corollary 2.2.4 *Let ϕ be a non-zero k-copositive map in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$. Then there exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely copositive map and γ is a p-trivial lifting of a $(k-1)$-copositive map in $B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$, for some $p \in \{1, ..., n\}$.*

Proof. If ϕ is k-copositive, using the same arguments in the proof of Theorem 2.2.2 for the matrix $\sum_{i,j=1}^{n} E_{ji} \otimes \phi(E_{ij})$, one obtains a decomposition $\sum_{i,j=1}^{n} E_{ji} \otimes \phi(E_{ij}) = \sum_{i,j=1}^{n} E_{ji} \otimes \psi(E_{ij}) + \sum_{i,j=1}^{n} E_{ji} \otimes \gamma(E_{ij})$, where ψ is a non-zero completely copositive map and γ is a $(k-1)$-copositive map which is a trivial lifting of a map in $B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$. $\square$

The decomposition in 2.2.2 and Corollary 2.2.4 is not unique. One can check this fact using the GC maps [YDT16].

Theorem 2.2.5 *Any non-zero k-positive (resp. k-copositive) map in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ is the sum of at most $(k-1)$ non-zero completely positive (resp. completely copositive) maps and a positive map which is the trivial lifting of a positive map in $B(M_{n-k+1}(\mathbb{C}), M_m(\mathbb{C}))$.*

Proof. For a k-positive linear map ϕ, repeatedly using Theorem 2.2.2 (respectively Corollary 2.2.4) until the remainder is a positive map. $\square$

The process in Choi decomposition may no longer apply to a general positive map ϕ even for $\phi \in B(M_2(\mathbb{C}), M_2(\mathbb{C}))$. Hence it may not necessarily give us an algorithm to decompose a positive map in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ as the sum of a completely positive map and a completely copositive map. The following example in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ is decomposable but one can not apply the aforementioned process in Choi decomposition.

Example 2.2.1 *Let ε be a real number and ω in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ be defined through its Choi matrix:*

$$C_\omega = \begin{pmatrix} 1 & 0 & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 1 \end{pmatrix},$$

Hence the map ω is given by

$$\omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \varepsilon(b+c) \\ \varepsilon(b+c) & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

For ω to be positive, it suffices to show for any vector $y = (y_1, y_2)^T \in \mathbb{C}^2$, the matrix

$$|y_1|^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y_1\overline{y_2} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} + y_2\overline{y_1} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} + |y_2|^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} |y_1|^2 & 2\varepsilon Re(y_1\overline{y_2}) \\ 2\varepsilon Re(y_1\overline{y_2}) & |y_2|^2 \end{pmatrix}$$

is positive. This is equivalent to the condition that $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$. For Choi decomposition, *using* $A_{11}^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, *we have*

$$C_\omega = \begin{pmatrix} 1 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & \varepsilon^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 1-\varepsilon^2 \end{pmatrix},$$

and using $A_{22}^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, *we have*

$$C_\omega = \begin{pmatrix} \varepsilon^2 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1-\varepsilon^2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In each of the two equations above, the last matrix corresponds to a linear map which is not positive. Meanwhile to decompose the map ω *as the sum of a completely positive map and a completely copositive map, one splits the original matrix as follows:*

$$C_\omega = \begin{pmatrix} 1/2 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

Obviously under this splitting, the second and the third matrix in the above equation correspond to a completely positive map ψ_1 *and a completely copositive map* ψ_2*, respectively, where* $\psi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{2} & \varepsilon b \\ \varepsilon c & \frac{d}{2} \end{pmatrix}$ *and* $\psi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{2} & \varepsilon c \\ \varepsilon b & \frac{d}{2} \end{pmatrix}$.

Naturally, we may ask if there is an algorithm to decompose positive linear maps in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$.

2.3 A Reduced Situation of $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$

In low dimensional cases such as $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$ and $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$, Woronowicz showed that every positive map is decomposable [Wo76]. In this section, we will show that in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$, although positive maps may not be decomposable, 2-positive maps are always decomposable. Let us start with a useful lemma. For any $p \in \{1, ..., m\}$, we assume that $\tilde{\chi}_p \in B(M_n(\mathbb{C}), M_m(\mathbb{C}))$ is the p-trivial lifting of a positive map $\chi \in B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$.

Lemma 2.3.1 *If χ is decomposable in $B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$, then its trivial lifting $\tilde{\chi}_p$ is also decomposable in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$.*

Proof. Given a decomposable map $\chi \in B(M_{n-1}(\mathbb{C}), M_m(\mathbb{C}))$, then $\chi = \chi^1 + \chi^2$, where χ^1 is completely positive and χ^2 is completely copositive. By Lemma 2.2.1, one obtains a completely positive map $\widetilde{\chi^1}_p$ and a completely copositive map $\widetilde{\chi^2}_p$ through p-trivial lifting of χ^1 and χ^2, respectively. By linearity of the trivial lifting, $\tilde{\chi}_p = (\widetilde{\chi^1 + \chi^2})_p = \widetilde{\chi^1}_p + \widetilde{\chi^2}_p$ is decomposable in $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$. $\square$

Theorem 2.3.2 *Every 2-positive or 2-copositive map ϕ in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable.*

Proof. Without loss of generality, we assume the 2-positive (resp.2-copositive) map ϕ is non-zero. In the case of $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$, the peel-off process yields that:

$$\phi = \psi + \tilde{\kappa}_p \textit{ for some } p \in \{1, 2, 3\}$$

where ψ is completely positive (resp. completely copositive) and $\tilde{\kappa}_p$ is a p-trivial lifting of a positive map $\kappa \in B(M_2(\mathbb{C}), M_3(\mathbb{C}))$. Since every positive map in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable in $B(M_2(\mathbb{C}), M_3(\mathbb{C}))$, by Lemma 2.3.1, the lifted map $\tilde{\kappa}_p$ is decomposable in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$. Hence, $\phi = \psi + \tilde{\kappa}_p$ is also decomposable. $\square$

Corollary 2.3.3 *Every indecomposable map in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is atomic.*

Proof. If a positive map in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is the sum of a 2-positive and a 2-copositive map, then it is decomposable by Theorem 2.3.2. $\square$

Corollary 2.3.4 *Under the dual cone correspondence, one can completely determine the set inclusion relations in* $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ *as follows:*

$$\begin{array}{ccccccccc} V_1[3,3] & \subsetneqq & T[3,3] & \subsetneqq & V_2[3,3] & \subsetneqq & V_3[3,3] & = & (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \| \\ P_1[3,3] & \supsetneqq & D[3,3] & \supsetneqq & P_2[3,3] & \supsetneqq & P_3[3,3] & \cong & (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \end{array}.$$

Proof. Theorem 2.3.2 shows that $D[3,3] \supseteq P_2[3,3]$. Further, the maps in Example 2.1.1 establish the inequalities in $P_1[3,3] \supseteq D[3,3] \supseteq P_2[3,3] \supseteq P_3[3,3]$. □

Remark: Note that the inclusion $T[3,3] \subsetneqq V_2[3,3]$ solves Conjecture 2.1.1.

2.4 Related Problems and Further Studies

Apparently, the next case to investigate is $(n, m) = (3, 4)$. Based on our experience, we pose the following Conjecture.

Conjecture 2.4.1 *There exist indecomposable* 2*-positive maps in* $B(M_3(\mathbb{C}), M_4(\mathbb{C}))$*.*

Conjecture 2.4.2 admits a dual version of finding PPTES in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$.

Conjecture 2.4.2 *There exist PPTES in* $M_3(\mathbb{C}) \otimes M_4(\mathbb{C})$ *of* $\mathrm{SN}(\rho) = 3$*.*

It remains a challenge to construct PPTES due to lack of structural results on the set $T[n,m] \backslash V_1[n,m]$. In chapter 3, we present a systematic method of constructing PPTES in $M_{n^2}(\mathbb{C}) \otimes M_{m^2}(\mathbb{C})$ of Schmidt number $r \in [1, \ldots, n \wedge m]$ but a "compact" example of PPTES as in Conjecture 2.4.2 is unknown. Generally, we do expect that the following holds especially when the dimensions of the underlying spaces are large enough.

Conjecture 2.4.3 *If* $nm \geq 12$*, then there exist PPTES in* $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ *of* $\mathrm{SN}(\rho) = n \wedge m$*. Or equivalently, there exist indecomposable* $(n \wedge m - 1)$*-positive maps in* $B(M_n(\mathbb{C}), M_m(\mathbb{C}))$*.*

Further, a guess on the complexity of determining the Schmidt number is given below.

Definition 2.4.1 *Let* $\{\Lambda_i\}_{i \in I}$ *be a collection of linear maps in* $B(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}), M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))$ *and* ρ *be a state in* $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$*. Assume that* $1 \leq s < n \wedge m$*.*

(a) A set $\{\Lambda_i\}_{i\in I}$ *is said to be a s-rule if* $\{\rho : \Lambda_i(\rho) \geq 0,\ \forall i \in I\} \subseteq V_s[n,m]$.

(b) A set $\{\Lambda_i\}_{i\in I}$ *is said to be a perfect s-rule if* $\{\rho : \Lambda_i(\rho) \geq 0,\ \forall i \in I\} = V_s[n,m]$.

For example, the collection $\{\tau_n \otimes id_m\}$ is a perfect 1-rule in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ when $(n,m) = (2,2),(2,3)$ while it is a 2-rule in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ when $(n,m) = (3,3)$. The existence of perfect s-rule is unknown in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ when $nm \geq 8$. Still we pose the following guess.

Conjecture 2.4.4 *If* $\{\Lambda_i\}_{i\in I}$ *is a perfect s-rule in* $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ *when* $nm \geq 8$*, then the cardinality* $|I| = \infty$.

If proven, Conjecture 2.4.4 gives a negative result saying that the determination of Schmidt number is complex in high dimensional spaces such that any finite numbered rule is insufficient.

Chapter 3

Schmidt Number of States under Local Projections

Most of this chapter is adopted from our joint paper [LYT16] which provides a repository of results on the Schmidt number of mixed states. The original paper is written in a physical style but it is rephrased into a mathematical style below.

3.1 Background and Current Status

The Schmidt number is a parameter characterizing quantum states. A quantum state is entangled if and only if its Schmidt number is greater than one. Entangled states play the fundamental role in quantum-information applications such as quantum computing and cryptography. One quantum state ρ is converted into another state σ under the physical environment of local operations and classical communications (LOCC) [CLMOW14]. In spite of the complex mathematical configuration of LOCC, the most basic operation in LOCC is the local projection P. P is a projection in $B(\mathcal{H})$ if $P^2 = P$ and $P^* = P$. We shall denote by P_A and P_B the projections in $B(\mathcal{H}_A)$ and $B(\mathcal{H}_B)$ respectively. Mathematically we have $\rho \to \sigma = (I_A \otimes P_B)\rho(I_A \otimes P_B)$, where I_A and I_B are the identities in $B(\mathcal{H}_A)$ and $B(\mathcal{H}_B)$ respectively. In this process the Schmidt number is entanglement monotone (non-increasing under LOCC) [TeHo00, Proposition 1], and the decrease of Schmidt number is decided by the local projection.

In analogy to the notions of Schmidt number and birank. we shall construct the notion of bi-Schmidt number in Eq. (3.1). Next we recall the definition of direct sum and tensor product of two quantum states, and obtain several preliminary results in Lemma 3.2.2 and 3.2.3.

The entanglement of the tensor product of two quantum states is invesitgated in Lemma 3.2.4. Next show in Lemma 3.2.7 that for any bipartite states ρ and σ with

$\mathrm{SN}(\sigma) \le \mathrm{SN}(\rho)$, the Schmidt number of the perturbation $\rho + \epsilon\sigma$ remains $\mathrm{SN}(\rho)$ for sufficiently small $\epsilon > 0$.

The results in this chapter are as follows. We will investigate how the local projection influences the Schmidt number of both bipartite and multipartite states.

(i) For bipartite states the investigation is carried out in Lemma 3.2.10 and 3.2.12. As an application we show that every 3×3 positive-partial-transpose (PPT) entangled state ρ is of Schmidt number 2 in Corollary 3.2.11. It provides an alternative proof for a conjecture in [SBL01]. We further show that the projected state σ can reach any integer smaller than the Schmidt number of ρ in Lemma 3.2.16. As an application of this result, we show in Theorem 3.2.17 that there exist bipartite PPT entangled states of any prescribed Schmidt number in a sufficiently large space. This is based on the preliminary results developed in Lemma 3.2.13 and Proposition 3.2.14. Moreover, it implies that, given any integer k, there exist indecomposable k-positive but not $(k+1)$ positive maps in $B(M_M(\mathbb{C}), M_N(\mathbb{C}))$ where M, N are sufficiently large. We also investigate when an entangled state can be projected onto a separable state in terms of their rank.

(ii) For multipartite states, we introduce the notion of expansion and coarse graining respectively in Definition 3.3.2 and 3.3.3. We investigate their relation to the Schmidt number of bipartite reduced density operators in Theorem 3.3.1 and Lemma 3.3.2. We further construct the notion of joint Schmidt number for multipartite states in Definition 3.3.4 and 3.3.5. We also restrict the joint Schmidt number of a multipartite pure state by the Schmidt numbers of its bipartite reduced density operators in Theorem 3.3.3. As an application, we show in Lemma 3.3.4 that any multipartite entangled PPT state with Schmidt number at least 3 when regarded as bipartite states, has rank at least 5.

In this Chapter, the following assumptions are made for simplicity. Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space with $\dim \mathcal{H}_A = M$ and $\dim \mathcal{H}_B = N$. Since the case $M = 1$ or $N = 1$ is trivial, we assume $2 \le M \le N$. Recall that ρ is a $M \times N$ state when $\operatorname{rank} \rho_A = M$ and $\operatorname{rank} \rho_B = N$, where ρ_A and ρ_B are the reduced states of the system A and B, respectively (see Definition 1.2.6). We shall work with bipartite quantum states ρ on $\mathcal{H}$. We shall write I_k for the identity $k \times k$ matrix. We denote by $\mathcal{R}(\rho)$ and $\ker \rho$ the range and kernel of a linear map ρ, respectively. From now on, unless stated otherwise, the states will not be normalized. We shall denote by $\{e_i : i = 1, \ldots, M\}$ and $\{f_j : j = 1, \ldots, N\}$ ONB of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. We say that ρ is PPT if $\rho^{\Gamma_A} \ge 0$. Otherwise ρ is NPT, i.e., ρ^{Γ_A} has at least one negative eigenvalue. We say that two bipartite states ρ and σ are equivalent under special local operations and classical communications (SLOCC) if there exists an invertible local operator (ILO) $A \otimes B$ such that $\rho = (A^* \otimes B^*)\sigma(A \otimes B)$ [DVC00]. In particular, they are locally equivalent when A and B are unitary matrices. It is easy to see that any ILO transforms PPT, entangled, or separable state into the same kind of states. We shall often use ILOs to simplify the density matrices of states. A subspace which contains no product state, is referred to as a completely entangled subspace (CES).

3.2 Schmidt Number of Bipartite States

Definition 3.2.1 *Let ρ be a bipartite state on $\mathcal{H}_A \otimes \mathcal{H}_B$. The birank of ρ is defined to be the pair of integers $(\operatorname{rank}\rho, \operatorname{rank}\rho^\Gamma)$. Further, assume that ρ is PPT. Similar to the birank, we define the pair of integers*

$$(\mathrm{SN}(\rho), \mathrm{SN}(\rho^\Gamma))$$

as the bi-Schmidt number, namely the BSN of ρ.

Unlike the birank, the BSN is defined for PPT states only because the Schmidt number is defined only for quantum states. Below is an application of BSN.

Lemma 3.2.1 *If ρ is a PPT state and* $\mathrm{SN}(\rho), \mathrm{SN}(\rho^\Gamma) \in \{1, 2\}$ *then* $\mathrm{SN}(\rho) = \mathrm{SN}(\rho^\Gamma)$.

Proof. Since ρ and its partial transpose ρ^Γ are simultaneously separable, so they have the same Schmidt number given that $\mathrm{SN}(\rho), \mathrm{SN}(\rho^\Gamma) \le 2$.

$\square$

Next we investigate the Schmidt number of the collective use of two quantum states. For this purpose we introduce two notions from quantum information.

The first notion is the direct sum of two spaces. It plays an important role in many quantum-information problems such as the distillability problem [ChDj16] and bipartite unitary operations [ChLi14, ChLi14ap, ChLi15]. We shall denote $V \oplus W$ as the ordinary direct sum of two matrices V and W, and $V \oplus_B W$ as the direct sum of V and W from the B side. The latter is called the B-direct sum, i.e., V and W respectively act on two subspaces $\mathcal{H}_A \otimes \mathcal{H}'_B$ and $\mathcal{H}_A \otimes \mathcal{H}''_B$ such that $\mathcal{H}'_B \perp \mathcal{H}''_B$. For example, the unnormalized state $(e_1 \otimes f_1)(e_1 \otimes f_1)^* + (e_1 \otimes f_2)(e_1 \otimes f_2)^*$ is the B-direct sum of $(e_1 \otimes f_1)(e_1 \otimes f_1)^*$ and $(e_1 \otimes f_2)(e_1 \otimes f_2)^*$.

The second notion from quantum information is the combination of different systems. Let $\rho_{A_iB_i}$ be an $M_i \times N_i$ state of rank r_i acting on the Hilbert space $\mathcal{H}_i = \mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i}$, $i = 1, 2$. The tensor product $\rho = \rho_{A_1B_1} \otimes \rho_{A_2B_2}$ is a state acting on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 = (\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}) \otimes (\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$. By switching the two middle factors, we can consider ρ as a composite bipartite state acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and $\mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$. In this case we shall write $\rho = \rho_{A_1A_2:B_1B_2}$ and call ρ a *flipped tensor product* of $\rho_{A_1B_1}$ and $\rho_{A_2B_2}$. Moreover, ρ is an $M_1M_2 \times N_1N_2$ state of rank not larger than r_1r_2. The above definition can be generalized to the flipped tensor product of N states $\rho_{A_iB_i}, i = 1, \ldots, N$. They form a bipartite state on the Hilbert space $\mathcal{H}_{A_1,\cdots,A_N} \otimes \mathcal{H}_{B_1,\cdots,B_N}$. For simplicity we denote the system A as $A_1, \cdots, A_N$ and denote B as $B_1, \cdots, B_N$.

For example, it is known that $\mathrm{SN}(\rho^{\otimes 2}) \in [\mathrm{SN}(\rho), \mathrm{SN}(\rho)^2]$, and $\mathrm{SN}(\rho^{\otimes 2})$ may reach any integer in the interval $[\mathrm{SN}(\rho), \mathrm{SN}(\rho)^2]$ when $\mathrm{SN}(\rho) = M$. An example is the two-qubit isotropic state [TeHo00, Fig. 1]. Now we have

Lemma 3.2.2 *Suppose $\rho = \alpha \oplus_B \beta$ where α and β are both bipartite quantum states. Then*
(i) $\mathrm{SN}(\rho) = \max\{\mathrm{SN}(\alpha), \mathrm{SN}(\beta)\}$.
(ii) $\mathrm{SN}(\rho^{\otimes n}) = \max\{\mathrm{SN}(\underbrace{\alpha \otimes_B \cdots \otimes_B \alpha}_{n}), \mathrm{SN}(\underbrace{\alpha \otimes_B \cdots \otimes_B \alpha}_{n-1} \otimes_B \beta), \cdots, \mathrm{SN}(\beta \otimes_B \underbrace{\alpha \otimes_B \cdots \otimes_B \alpha}_{n-1}), \cdots, \mathrm{SN}(\underbrace{\beta \otimes_B \cdots \otimes_B \beta}_{n})\}$.

Proof. (i) By definition we have $\mathrm{SN}(\rho) \leq \max\{\mathrm{SN}(\alpha), \mathrm{SN}(\beta)\}$. On the other hand we can project ρ onto α and β by local projections. Since the Schmidt number is an entanglement monotone we have $\mathrm{SN}(\rho) \geq \max\{\mathrm{SN}(\alpha), \mathrm{SN}(\beta)\}$.

(ii) The assertion follows from (i). This completes the proof. □

We generalize the Lemma as follows. It is known that any quantum physical operation can be expressed as a completely positive (CP) map $(\rho) := \sum_i P_i \rho P_i^*$ where $\sum_i P_i^* P_i \leq I$, where P_i is a projection. If the equality holds then the operation is a completely positive trace-preserving (CPTP) map, namely a quantum channel. We construct the relation between quantum operation and Schmidt number.

Lemma 3.2.3 *Suppose ρ is a bipartite state, and $\Lambda(\cdot) = \sum_i P_i(\cdot)P_i^*$ is a quantum operation such that $(I_A \otimes \Lambda)\rho = \rho$. Then $\mathrm{SN}(\rho) = \max_i\{\mathrm{SN}(\rho_i)\}$ where $\rho_i = (I_A \otimes P_i)\rho(I_A \otimes P_i^*)$.*

Proof. By definition we have $\mathrm{SN}(\rho) \leq \max_i\{\mathrm{SN}(\rho_i)\}$. Since the Schmidt number is an entanglement monotone we have $\mathrm{SN}(\rho) \geq \max_i\{\mathrm{SN}(\rho_i)\}$. This completes the proof. $\square$

If the channel is $\Lambda(\cdot) = P(\cdot)P^* + (I-P)(\cdot)(I-P^*)$ where P is a projection, then Lemma 3.2.3 reduces to Lemma 3.2.2. Finding out the states ρ satisfying the hypothesis of Lemma 3.2.3 is an interesting question. For example, we can assume ρ as the quantum-classical separable state $\rho = \sum_i p_i \rho_i \otimes e_i e_i^*$ [CCMV11].

The following Lemma investigates the entanglement of the tensor product of two quantum states.

Lemma 3.2.4 *Let the integers $m_1, n_1, m_2, n_2 \in \{2, 3\}$, $m_1 + n_1 < 6$ and $m_2 + n_2 < 6$. Suppose ρ_1 and ρ_2 are $m_1 \times n_1$ and $m_2 \times n_2$ states in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}$ and $\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2}$, respectively. $\rho_1 \otimes \rho_2$ is a bipartite state w.r.t the bi-partition $A_1A_2 : B_1B_2$.*
(i) If either of the two states ρ_1 and ρ_2 is entangled, then $\rho_1 \otimes \rho_2$ is a NPT state.
(ii) Conversely, if $\rho_1 \otimes \rho_2$ is a PPT state, then both ρ_1 and ρ_2 are separable states.

Proof. (i) Assume that ρ_1 is entangled. By the Peres-Horodecki criterion [Pe96], we have $\rho_1^{\Gamma_{A_1}}$ is not positive semidefinite hence the least eigenvalue of $\rho_1^{\Gamma_{A_1}}$ is negative. On the other hand, as a matrix in $M_{m_1}(M_{n_1}(\mathbb{C}))$, ρ_1 and $\rho_1^{\Gamma_{A_1}}$ share the same block diagonal elements. If all eigenvalues of $\rho_1^{\Gamma_{A_1}}$ are negative, then the block diagonal elements are zero matrices since $\rho_1 \geq 0$ and $-\rho_1^{\Gamma_{A_1}} \geq 0$. Hence there exists a positive eigenvalue in the spectrum of $\rho_1^{\Gamma_{A_1}}$. Since the eigenvalues of $(\rho_1 \otimes \rho_2)^{\Gamma_{A_1A_2}} = \rho_1^{\Gamma_{A_1}} \otimes \rho_2^{\Gamma_{A_2}}$ are the pairwise products of eigenvalues of $\rho_1^{\Gamma_{A_1}}$ and $\rho_2^{\Gamma_{A_2}}$, there exists a negative eigenvalue in the spectrum of $(\rho_1 \otimes \rho_2)^{\Gamma_{A_1B_1}}$.
(ii) follows (i) immediately. This completes the proof. $\square$

The Lemma shows that the entanglement of the tensor product implies the entanglement of at least one state in the tensor product. On the other hand, if $\rho_1 + \rho_2$ is a separable state then ρ_1 and ρ_2 may be both entangled. An example is $\rho_1 = \alpha_+\alpha_+^*$ and $\rho_2 = \alpha_-\alpha_+^*$ where $\alpha_\pm = (e_1 \otimes e_1) \pm (e_2 \otimes e_2)$. This is different from (ii) which works for the tensor product of two states. Moreover if we want to construct PPT entangled states using the tensor product of two PPT entangled states by Lemma 3.2.4, then ρ_1 and ρ_2 have to be $M \times N$ PPT entangled states where $M, N \geq 3$.

As another application of Schmidt rank, we introduce a subspace containing only highly entangled states [CMW08].

Definition 3.2.2 *A subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$ is said to be a k-CES ($k \leq min\{m,n\}$) if it contains no nonzero Schmidt rank l vectors for $l \leq k$.*

Definition 3.2.3 *Consider a multipartite quantum system $\mathcal{H} = \otimes_{i=1}^m \mathcal{H}_i$ with m parties. An orthogonal product basis (PB) is a set S of pure orthogonal product states spanning a proper subspace $\mathcal{H}_S$ of $\mathcal{H}$. An unextendible product basis (UPB) is an PB whose complementary subspace $\mathcal{H}_S^\perp$ contains no product state.*

For example, if the range of a bipartite state is 1-completely entangled then the state is entangled. This is how the PPT entangled states by unextendible product bases are constructed [BDMSST99].

Lemma 3.2.5 *If ρ is a bipartite quantum state whose $R(\rho)$ is a k-CES, then* $\mathrm{SN}(\rho) \geq k+1$.

Proof. Since the range $R(\rho)$ is the linear span of the vectors $\{\xi_i\}$ in any decomposition $\rho = \sum_i \xi_i^* \xi_i$, we have $\mathrm{SN}(\rho) \geq k+1$ by Definition 1.2.7 and Definition 3.2.2. □

The Lemma gives a sufficient condition such that ρ is entangled. The condition is not necessary. An example is the two-qubit state $(e_1 \otimes f_1)(e_1 \otimes f_1)^* + ((e_1 \otimes f_1) + (e_2 \otimes f_2))((e_1 \otimes f_1) + (e_2 \otimes f_2))^*$. One can easily show that the state is entangled and its range is not 1-completely entangled. On the other hand, the construction of k-CES seems hard. Hitherto most results shows that estimating the Schmidt number is a hard problem. The following result from [TeHo00] provides a method for the estimation in terms of the maximally entangled states.

Lemma 3.2.6 *For any density matrix ρ with $M = N$ and Schmidt number k, we have*

$$\max_{\xi_M} (\xi_M^* \rho \xi_M) \leq \frac{k}{M}, \tag{3.1}$$

where we maximize over $M \times M$ bipartite maximally entangled states $\xi_M \triangleq \sum_{i=1}^M (e_i \otimes f_i) \otimes (e_i \otimes f_i)^*$ where $\{e_i\}_{i=1}^M$ and $\{f_i\}_{i=1}^M$ are ONB of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

An equivalent statement is presented in [GMS15, Proposition 2.4.12]. That is if $(\xi_M^* \rho \xi_M) > \frac{k}{N}$ for some maximally entangled state ξ_M then $\mathrm{SN}(\rho) > k$. This result can be used to infer the Schmidt number of quantum states. For example let us consider the mixed state $\rho = \sum_i p_i \xi_i \xi_i^*$, where ξ_1 has the maximum Schmidt rank.

The greater p_1 is, the greater $(\xi_M^* \rho \xi_M)$ becomes. Then Lemma 3.2.6 shows that the Schmidt number of ρ also increases.

Next we show that the Schmidt number is stable under perturbation.

Lemma 3.2.7 *For any bipartite states ρ and σ with $\mathrm{SN}(\sigma) \leq \mathrm{SN}(\rho)$, the Schmidt number of the perturbation $\rho + \epsilon\sigma$ remains $\mathrm{SN}(\rho)$ for sufficiently small $\epsilon > 0$.*

Proof. For any $l \geq \mathrm{SN}(\rho) \geq \mathrm{SN}(\sigma)$, we have $\mathrm{Tr}(\rho C_\phi^t) \geq 0 \ \forall \phi \in P_l$ and $\mathrm{Tr}(\sigma C_\phi^t) \geq 0 \ \forall \phi \in P_l$ by equation (1.1). Therefore $\mathrm{Tr}((\rho + \epsilon\sigma) C_\phi^t) = \mathrm{Tr}(\rho C_\phi^t) + \epsilon \,\mathrm{Tr}(\sigma C_\phi^t) \geq 0 \ \forall \phi \in P_l$ for any non-negative ϵ. On the other hand, taking $l = \mathrm{SN}(\rho) - 1$, there exists a positive map $\psi \in P_l$ such that $\mathrm{Tr}(\rho C_\psi^t) < 0$ by equation (1.1). Choosing a sufficiently small ϵ, we also have $\mathrm{Tr}((\rho + \epsilon\sigma) C_\psi^t) = \mathrm{Tr}(\rho C_\psi^t) + \epsilon \,\mathrm{Tr}(\sigma C_\psi^t) < 0$. Hence by equation (1.1) we have $\mathrm{SN}(\rho + \epsilon\sigma)$ remains $\mathrm{SN}(\rho)$ for sufficiently small ϵ. $\square$

3.2.1 The Local Projections

First we review and construct a few results on linear algebra used throughout the following sections. We have seen in Definition 1.2.7 that computing the Schmidt number of a quantum state requires the investigation of all decompositions of the state. The following result provides the closed formula for the decomposition [HJW93].

Lemma 3.2.8 *Let ρ be a quantum state and the spectral decomposition $\rho = \sum_i p_i \alpha_i \alpha_i^*$ such that $p_i > 0$ and the α_i are pairwise orthonormal states. Then any decomposition $\rho = \sum_{j=1}^m q_j \beta_j \beta_j^*$ with $q_j > 0$ satisfies $\sqrt{q_j}\beta_j = \sum_i u_{ij}\sqrt{p_i}\alpha_i$ for an order-m unitary matrix $[u_{ij}]$.*

The Lemma will be used in the proof of Lemma 3.2.10 studying the Schmidt number of quantum states and their projections. The next result is used for detecting the Schmidt number of bipartite states in Lemma 3.2.13.

If $\xi = \sum_i \alpha_i \otimes \beta_i \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\gamma \in \mathcal{H}_A$, then we abuse the notation and denote by $\gamma^*\xi$ the vector $\sum_i (\gamma^*\alpha_i)\beta_i$ in $\mathcal{H}_B$.

Lemma 3.2.9 *Suppose ξ and η are two bipartite states in $\mathcal{H}_A \otimes \mathcal{H}_B$. There exists a nonzero state $\gamma \in \mathcal{H}_A$ or $\mathcal{H}_B$ such that the two states $\gamma^*\xi$ and $\gamma^*\eta$ in $\mathcal{H}_B$ or $\mathcal{H}_A$ are proportional, and at least one of them is nonzero.*

Proof. Suppose $\{\alpha_j\}_{j=1,\cdots,M}$ and $\{\beta_j\}_{j=1,\cdots,N}$ are respectively two orthonormal basis in $\mathcal{H}_A$ and $\mathcal{H}_B$. Without loss of generality, we assume that ξ is not parallel to η. We write the Schmidt decomposition as $\xi = \sum_{j=1}^{L} c_j(\alpha_j \otimes \beta_j)$ where $c_j \neq 0$, $L \leq M \leq N$, and $\eta = \sum_{j=1}^{M} \sum_{k=1}^{N} d_{jk}(\alpha_j \otimes \beta_k)$ [NC10]. If some $d_{jk} \neq 0$ when $L < j$ or $L < k$, then we choose $\gamma = \alpha_j$ or $\gamma = \beta_k$, and the assertion holds. If all $d_{jk} = 0$ when $L < j$ or $L < k$, we can find two complex number x, y such that the nonzero state $x\xi + y\eta$ has Schmidt number strictly less than L. Choose $\gamma \in \operatorname{span}\{\alpha_1, \cdots, \alpha_L\}$ and $\gamma^*(x\xi + y\eta) = 0$. Then the two states $\gamma^*\xi$ and $\gamma^*\eta$ in $\mathcal{H}_B$ are proportional, and $\gamma^*\xi$ is nonzero. So the assertion holds. This completes the proof. $\square$

Note that the space in which the state γ belongs to cannot be fixed. An example is that $\xi = (e_1 \otimes f_1) + (e_2 \otimes f_2)$ and $\eta = (e_1 \otimes f_2) + (e_2 \otimes f_3)$. One can show that no $\gamma \in \mathcal{H}_A$ satisfies the assertion. On the other hand one can choose $\gamma = f_1 \in \mathcal{H}_B$.

In this section we investigate the Schmidt number of bipartite states under local projections. Bipartite entangled states are the fundamental resources in quantum computing and cryptography. For this purpose bipartite states are converted into maximally entengled states with a smaller Schmidt number under local projections asymptotically. This is the well-known entanglement distillation or purification [BDSW96]. Next, bipartite states are entangled if and only if they have Schmidt number greater than one. Deciding whether a state is entangled is the well-known separability problem. One may detect the entanglement by locally projecting the target state onto another state with smaller dimensions. The local projections play important roles in both issues. We begin by proposing a preliminary Lemma on the Schmidt number and local projections.

Lemma 3.2.10 *Let ρ be an $M \times N$ entangled state, $k \in [1, M-1]$ an integer, P a matrix of rank $M-k$, and $\sigma = (P \otimes I_B)\rho(P^* \otimes I_B)$ the projected state. Then*
(i)

$$\max\{1, \mathrm{SN}(\rho) - k\} \leq \mathrm{SN}(\sigma) \leq \min\{\mathrm{SN}(\rho), M-k\}. \tag{3.2}$$

(ii) We have $\rho = \sum_j \xi_j \xi_j^*$, where $\xi_j = \sum_{l=1}^{\mathrm{SN}(\sigma)} \alpha_{j,l} \otimes \beta_{j,l} + \sum_{i=1}^{k} \nu_i \otimes \mu_{j,i}$, $\mathcal{R}(P) = \operatorname{span}\{\alpha_{j,l}\}$, $\nu_i \perp \nu_j$, and $\nu_i \perp P$ for all i, j.
(iii) If $\mathrm{SN}(\rho) = M$, then $\mathrm{SN}(\sigma) = M-k$.

Below we further assume that ρ is PPT. Then

(iv)

$$\max\{1, \mathrm{SN}(\rho^\Gamma) - k\} \le \mathrm{SN}(\sigma^\Gamma) \le \min\{\mathrm{SN}(\rho^\Gamma), M - k\}. \tag{3.3}$$

(v) If $k = \mathrm{SN}(\sigma) = 1$, *then* $\mathrm{SN}(\rho) = \mathrm{SN}(\rho^\Gamma) = 2$.
(vi) If $k = \min\{\mathrm{SN}(\rho), \mathrm{SN}(\rho^\Gamma)\} - s$, *and* $\mathrm{SN}(\rho) \ne \mathrm{SN}(\rho^\Gamma)$, *then* $\max\{\mathrm{SN}(\sigma), \mathrm{SN}(\sigma^\Gamma)\} \ge s + 1$.
(vii) If $k = M - 2$ *or* $M - 1$, *then* $\mathrm{SN}(\sigma) = \mathrm{SN}(\sigma^\Gamma) \in \{1, 2\}$.
(viii) If $\mathrm{SN}(\rho) = \mathrm{SN}(\rho^\Gamma)$, *then* $\mathrm{SN}(\sigma) - \mathrm{SN}(\sigma^\Gamma) \in [-k, k]$.

Proof. (i) Since the Schmidt number of quantum states is invariant up to local invertible operators, we may assume that P is a projection. Let $P = \sum_{i=1}^{M-k} \kappa_i \kappa_i^*$ and $\{\kappa_1, \cdots, \kappa_M\}$ an ONB of $\mathcal{H}_A$. Let $\rho = \sum_j \xi_j \xi_j^*$ where $\xi_j = \sum_{i=1}^M \kappa_i \otimes u_{ij}$ and u_{ij} are non-normalized vectors. We have

$$\begin{aligned} \alpha_j &:= (P \otimes I_B)\xi_j \\ &= \sum_{i=1}^{M-k} \kappa_i \otimes u_{ij} := \xi_j - \beta_j, \end{aligned} \tag{3.4}$$

where

$$\beta_j = \sum_{i=M-k+1}^{M} \kappa_i \otimes u_{ij}. \tag{3.5}$$

Using Lemma 3.2.8 we may assume that $|\alpha_j\rangle$ are pairwise orthogonal, and we do not change the expression of ρ since there is no confusion. Since $\sigma = \sum_j \alpha_j \alpha_j^*$, we can find a unitary matrix $W = [w_{jl}]$ such that for any k the pure state $\sum_j w_{jl} \alpha_j$ has Schmidt rank at most $\mathrm{SN}(\sigma)$. Hence

$$\begin{aligned} \rho &= \sum_j (\alpha_j + \beta_j)(\alpha_j + \beta_j)^* \\ &= \sum_l \Big(\sum_j w_{jl}(\alpha_j + \beta_j) \Big) \Big(\sum_j \overline{w_{jl}} (\alpha_j + \beta_j)^* \Big). \end{aligned} \tag{3.6}$$

The definition of Schmidt number and (3.5) imply that $\mathrm{SN}(\rho) \le \mathrm{SN}(\sigma) + k$. Since σ is nonzero we always have $\mathrm{SN}(\sigma) \ge 1$. So we have proved the lower bound in (3.2).

On the other hand, it is known that the Schmidt number is monotone decreasing under the local operations and classical communications [TeHo00]. So $\mathrm{SN}(\sigma) \le \mathrm{SN}(\rho)$. Besides, the inequality $\mathrm{SN}(\sigma) \le M - k$ follows from the fact that P has rank $M - k$. We have proved (i).

(ii) It suffices to prove $\mathcal{R}(P) = \mathrm{span}\{\alpha_{j,l}\}$. The inclusion $\mathcal{R}(P) \supseteq \mathrm{span}\{\alpha_{j,l}\}$ is evident. If the inclusion is strict, then $\mathrm{rank}\, P > \mathrm{rank}\, \sigma_A$.

On the other hand Since $(P \otimes I_B)\rho(P^* \otimes I_B) = \sigma$, we have $P\rho_A P^* = \sigma_A$. Since $\mathrm{rank}\, \rho_A = M$ we have $\mathrm{rank}\, P = \mathrm{rank}\, \sigma_A$. We have a contradiction and thus $\mathcal{R}(P) = \{\alpha_{j,l}\}$.

(iii) The assertions both follow from the proof of (i).

(iv) The assertion follows from (i) by replacing ρ by ρ^Γ.

(v) Since $k = 1$ and $\mathrm{SN}(\sigma) = 1$, (i) implies $1 \le \mathrm{SN}(\rho) \le 2$, and (iv) implies $1 \le \mathrm{SN}(\rho^\Gamma) \le 2$. Since ρ and ρ^Γ are both separable or not, we have proved the assertion.

(vi) The assertion follows from (i).

(vii) The assertion follows from (i).

(viii) The assertion follows by summing up (3.2) and minus (3.3). This completes the proof. $\square$

By checking the proof of Lemma 3.2.10, one can show that it also holds when $M > N$. In Lemma 3.2.10 (i), the Schmidt number of the $M \times N$ bipartite state ρ is dominated by the sum of the Schmidt number of the projected states σ plus the dimension of the kernel of the projection. In Lemma 3.2.10 (ii) if $\mathrm{SN}(\rho) \le k$ then $\xi_j = \sum_{i=1}^k \nu_i \otimes \mu_{i,j}$. It is impossible unless $k = M$. So the last inequality in (3.2) may be strict. An example is the 3×3 Werner state ρ and $k = 1$. Since any σ is a 2×3 separable state, we have $\mathrm{SN}(\sigma) = 1 < \mathrm{SN}(\rho) = 2 = M - k$. The first inequality in (3.2) may be also strict. First we give an example of NPT ρ and $M = N = 3$. An example is the antisymmetric state $\rho = \sum_{j,k=1,j<k}^3 (e_j \otimes f_k - e_k \otimes f_j)(e_j \otimes f_k - e_k \otimes f_j)^*$. Up to ILOs we may assume the projection $P = e_1 e_1^* + e_2 e_2^* + (a e_1 + b e_2) e_3^*$ where a, b are complex numbers. Then $(P \otimes I_2)\rho(P^* \otimes I_2)$ is an NPT two-qubit state for any a, b. So it is entangled, and $\mathrm{SN}(\rho) = \mathrm{SN}(\sigma) = 2$.

Corollary 3.2.11 *Let ρ be a 3×3 state. Then*
(i) every PPT entangled ρ is of Schmidt number 2;
(ii) every state ρ of Schmidt number 3 is NPT. Moreover, for any matrix $P, Q \in$

$M_3(\mathbb{C})$ with rank(P) = rank$(Q) = 2$, the projected states $(P \otimes I_3)\rho(P^* \otimes I_3)$ or $(I_3 \otimes Q)\rho(I_3 \otimes Q^*)$ are NPT states.

Proof. (i) This assertion follows Lemma 3.2.10 (i), in which we set $M = N = 3$ and $k = 1$. Then we have $\mathrm{SN}(\rho) \leq \mathrm{SN}(\sigma) + 1$. Note that σ is a 2×3 PPT state which is also separable by Peres-Horodecki criterion.

(ii) The first assertion follows easily from (i). WLOG, assume that the projected states $\sigma = (P \otimes I_3)\rho(P^* \otimes I_3)$ is a PPT state. So σ is a separable state, hence it violates the inequality $\mathrm{SN}(\sigma) \geq \mathrm{SN}(\rho) - k = 2$. □

The projected states may not be NPT even if the original state is NPT. For example, for any rank-one P the state $(P \otimes I_B)\rho(P^* \otimes I_B)$ is a separable state. It is an open problem to find out when the projected state is NPT, and it relates to the well-known distillability problem which will be introduced in Chapter 4.

Below is an example of PPT state, where $k = 1$ and $\mathrm{SN}(\rho) = 2$. Note that these two states also saturate the last equality in (3.2).

Example 3.2.1 *Let $\rho = \varsigma \oplus \zeta$ be a PPT entangled state, where ς and ζ are both 3×3 PPT entangled states, $\mathcal{R}(\varsigma_A) = \mathcal{R}(\varsigma_B) = \mathrm{span}\{e_1, e_2, e_3\}$ and $\mathcal{R}(\zeta_A) = \mathcal{R}(\zeta_B) = \mathrm{span}\{e_4, e_5, e_6\}$. It follows from Lemma 3.2.2 and Corollary 3.2.11 that $\mathrm{SN}(\rho) = \mathrm{SN}(\varsigma) = \mathrm{SN}(\zeta) = 2$.*

Let P_A be a projection of rank five on $\mathcal{H}_A$. We can express P_A as $P_A = \sum_{i=1}^{6} \alpha_i e_i^*$, where $\alpha_1, \cdots, \alpha_6$ span a 5-dimensional subspace in $\mathbb{C}^6$. Hence either $\alpha_1, \alpha_2, \alpha_3$ or $\alpha_4, \alpha_5, \alpha_6$ span a 3-dimensional subspace in $\mathbb{C}^6$. Let $\sigma = (P_A \otimes I_B)\rho(P_A \otimes I_B)$. We have

$$\begin{aligned} \sigma &= \left(\sum_{i=1}^{3} \alpha_i e_i^*\right)_A \varsigma \left(\sum_{i=1}^{3} e_i \alpha_i^*\right)_A \\ &\oplus_B \left(\sum_{i=4}^{6} \alpha_i e_i^*\right)_A \zeta \left(\sum_{i=4}^{6} e_i \alpha_i^*\right)_A . \end{aligned} \tag{3.7}$$

So either the first state or the second state in (3.7) is still a 3×3 PPT entangled state. It follows from Lemma 3.2.2 and Corollary 3.2.11 that $\mathrm{SN}(\sigma) = 2 = \mathrm{SN}(\rho)$.

In Lemma 3.2.10 (iii), one can generate quantum states of Schmidt number $M - k$ using rank $M - k$ projections from a Schmidt number M state. The converse of (iii) does not hold. An example is the normalized antisymmetric projection on the

3×3 subspace. This is an entangled state. Further we propose an example of a separable state. Consider a 2×3 PPT state ρ with any rank 1 projection, we have $\mathrm{SN}(\rho) = 1 < M$ and $\mathrm{SN}(\sigma) = 1 = M - k$.

Lemma 3.2.10 provides an alternative proof for Conjecture 2.1.1, see the Corollary below.

Next we consider the relation between the Schmidt numbers of the two tensors of the two copies of a bipartite state and the two copies of its projected state.

Lemma 3.2.12 *If ρ and σ are as introduced in Lemma 3.2.10, then*

$$\mathrm{SN}(\sigma^{\otimes 2}) \le \min\{\mathrm{SN}(\rho^{\otimes 2}), (M-k)^2\}, \tag{3.8}$$

$$\mathrm{SN}(\rho^{\otimes 2}) \le \mathrm{SN}(\sigma)^2 + 2k\,\mathrm{SN}(\sigma) + k^2. \tag{3.9}$$

Proof. First we prove (3.8). Since $\sigma = (P \otimes I_B)\rho(P^* \otimes I_B)$, we can project $\rho^{\otimes 2}$ onto $\sigma^{\otimes 2}$. Hence $\mathrm{SN}(\sigma^{\otimes 2}) \le \mathrm{SN}(\rho^{\otimes 2})$. It follows from (3.2) that $\mathrm{SN}(\sigma) \le M - k$. So σ is the convex sum of pure states of Schmidt rank at most $M - k$. So $\sigma^{\otimes 2}$ is the convex sum of pure states of Schmidt rank at most $(M-k)^2$. We have $\mathrm{SN}(\sigma^{\otimes 2}) \le (M-k)^2$. So (3.8) holds. Next (3.9) follows from the fact $\mathrm{SN}(\rho) \le \mathrm{SN}(\sigma) + k$, which is from Lemma 3.2.10 (i) and (ii). This completes the proof. □

The Lemma shows that the Schmidt number of the tensor product of the two copies of the same state is bounded by that of the tensor product of its projected states. One may similarly extend the Lemma to the tensor product of many copies of the same states. We further investigate the Schmidt number of the tensor product of different mixed states. The following result shows that such Schmidt number may be greater than the Schmidt number of each of them.

In the following Lemma, the subscript A_1A_2 in $a_{iA_1A_2}$ indicates that a_i is a vector in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$. $\rho_i = \xi_i\xi^*$ denotes the *ith* summand in $\rho = \sum_i \xi_i\xi_i^*$.

Lemma 3.2.13 *Let $\rho = \alpha_{A_1B_1} \otimes \beta_{A_2B_2}$ be a bipartite state on the system A_1A_2 and B_1B_2.*

(i) If neither of the range of the states $\alpha_{A_1B_1}$ and $\beta_{A_2B_2}$ contains any product state, then $\mathrm{SN}(\rho) > 2$, and any decomposition of ρ consists of pure states of Schmidt rank at least three.

(ii) In (i) if $\mathrm{SN}(\rho) = 3$, then $\rho = \sum_i \xi_i\xi_i^*$ where

$$\xi_i = a_{iA_1A_2} \otimes b_{iB_1B_2} + c_{iA_1A_2} \otimes d_{iB_1B_2} + g_{iA_1A_2} \otimes h_{iB_1B_2}, \tag{3.10}$$

is a bipartite state of Schmidt number three. For any i, the spaces $\mathcal{R}((\rho_i)_{A_1A_2})$ and $\mathcal{R}((\rho_i)_{B_1B_2})$ both have no product state.
(iii) If $\alpha_{A_1B_1}$ and $\beta_{B_1B_2}$ are both two-qutrit PPT entangled states of rank four, then $\mathrm{SN}(\rho) = 4$.

Proof. Since the range of the state $\alpha_{A_1B_1}$ does not contain any product state, $\alpha_{A_1B_1}$ is entangled. So ρ is also entangled and has Schmidt number at least two. Since the range of $\alpha_{A_1B_1}$ does not contain any product state, the pure state in any decomposition of ρ is a bipartite entangled state.

(i) We prove by contradiction. Suppose there is a decomposition of ρ containing a Schmidt-rank-two bipartite pure entangled state, i.e., $\rho = \sum_i \xi_i\xi_i^*$ where

$$\xi_1 = a_{1A_1A_2} \otimes b_{1B_1B_2} + c_{1A_1A_2} \otimes d_{1B_1B_2}. \tag{3.11}$$

It follows from Lemma 3.2.9 that there exists a nonzero state $\gamma \in \mathcal{H}_{A_1}$ (or $\mathcal{H}_{A_2}$) such that the two states γ^*a_1 and γ^*c_1 in $\mathcal{H}_{A_2}$ (or $\mathcal{H}_{A_1}$) are proportional, and one of them is nonzero. Hence $\gamma^*\xi_1$ is a product state of the system A_2 (or A_1) and B_1B_2. By tracing out system A_1B_1 (or A_2B_2), we obtain that the range of $\beta_{A_2B_2}$ (or $\alpha_{A_1B_1}$) contains a product state. It is a contradiction with the assumptions. So we have $\mathrm{SN}(\rho) > 2$, and any decomposition of ρ consists of pure states of Schmidt rank at least three.

(ii) The first assertion follows from (i). Using (3.12) we shall regard a_i, c_i, g_i as an arbitrary basis of $\mathcal{R}((\rho_i)_{A_1A_2})$, and b_i, d_i, h_i as an arbitrary basis of $\mathcal{R}((\rho_i)_{B_1B_2})$. To prove the second assertion, it suffices to show that for any i, the states $a_i, b_i, c_i, d_i, g_i, h_i$ all have Schmidt number greater than one. We have three cases.

In the first case, we assume that a_i, c_i and g_i are product states. Let $a_i = w_1 \otimes w_2$, $c_i = x_1 \otimes x_2$ and $g_i = y_1 \otimes y_2$. The second assertion is trivial when for $j = 1$ or 2, two of the states w_j, x_j and y_j are proportional, or all of the three states are linearly independent. The only unsolved case is that for $j = 1$ and 2, any two of w_j, x_j and y_j are linearly independent and all of the three states are linearly dependent. According to Lemma 3.2.9, there exists a nonzero state $\gamma \in \mathcal{H}_{B_1}$ or $\mathcal{H}_{B_2}$ such that the two states γ^*d and γ^*h in $\mathcal{H}_{B_2}$ or $\mathcal{H}_{B_1}$ are proportional, and one of them is nonzero. Let $z \perp w_1$ or w_2, and z is not orthogonal to y_1, z_1 or y_2, z_2. Then $z^*(\gamma^*\psi_i)$ is a product state. We trace out $\rho_{A_1B_1}$ by using the state $z \otimes \gamma$ as a state in the trace. Then one can show the second assertion, since the range of the state $\alpha_{A_1B_1}$ and $\beta_{A_2B_2}$ does not contain any product state.

Next we assume that a_i and c_i are product states, and g_i is an entangled state. If

$g_i + xa_i + yc_i$ is a product state for some complex numbers x, y then we have proved the assertion in the first case. So $g_i + xa_i + yc_i$ is an entangled state for any x, y. It implies that there is a state $z \in \mathcal{H}_{A_1}$ or $\mathcal{H}_{A_2}$ such that $z^* g_i \neq 0$ and $z^* a_i = z^* c_i = 0$. By tracing out one of $\alpha_{A_1 B_1}$ and $\beta_{A_2 B_2}$, we can obtain that the range of the other state contains product states. It is a contradiction with the assumption.

Thirdly we assume that a_i is a product state, and c_i and g_i are both entangled states. If $g_i + xa_i + yc_i$ is a product state for some complex numbers x, y then we have proved the assertion in the last two cases. So $g_i + xa_i + yc_i$ is an entangled state for any x, y. One can similarly show that $c_i + xa_i + yg_i$ is an entangled state for any x, y. Lemma 3.2.9 implies that there is a state $\gamma \in \mathcal{H}_{B_1}$ or $\mathcal{H}_{B_2}$ such that the two states $\gamma^* d_i$ and $\gamma^* h_i$ in $\mathcal{H}_{B_2}$ or $\mathcal{H}_{B_1}$ are proportional, and one of them is nonzero. We have $\gamma^* \psi_i = a_i \otimes \gamma^* b_i + p_i \otimes q_i$, where p_i is the linear combination of c_i and g_i. So p_i is an entangled state. We can find a state $q \in \mathcal{H}_{A_1}$ or $\mathcal{H}_{A_2}$ such that $q^* a_i = 0$ and $q^* p_i \neq 0$. So $\mathcal{R}(\alpha_{A_1 B_1})$ or $\mathcal{R}(\beta_{A_2 B_2})$ contains a product state $q^* p_i \otimes q_i$. It is a contradiction with the assumption.

One can similarly prove that the spaces $\mathcal{R}((\rho_i)_{B_1 B_2})$ have no product state for any i by exchanging the systems $A_1 A_2$ and $B_1 B_2$.

(iii) It is known that neither of the range of the states $\alpha_{A_1 B_1}$ and $\beta_{A_2 B_2}$ contains any product state [ChDj13]. Further we can choose that a_i and c_i have Schmidt rank two, because $\mathcal{R}(\rho_{A_1 A_2})$ is a 3-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$. Next if there is a state $\nu \in \mathcal{H}_{A_1}$ or $\mathcal{H}_{A_2}$ orthogonal to a_i, c_i and g_i at the same time, then $\mathcal{R}(\rho_{A_1 A_2}) \subset \nu^\perp \otimes \mathbb{C}^3$. So $\mathcal{R}(\rho_{A_1 A_2})$ contains a product state and it is a contradiction with (ii). Hence there is no state orthogonal to a_i, c_i and g_i at the same time. It implies that if there is a state $\nu \in \mathcal{H}_{A_1}$ or $\mathcal{H}_{A_2}$ orthogonal to a_i, c_i, then there is a product state in $\mathcal{R}(\alpha_{A_2 B_2})$ or $\mathcal{R}(\beta_{A_1 B_1})$. It is in contradiction to (ii). So such ν does not exist. We shall use these facts below.

It follows from Lemma 3.2.9 that there exists a nonzero state $\gamma \in \mathcal{H}_{B_1}$ or $\mathcal{H}_{B_2}$ such that the two states $\gamma^* d_i$ and $\gamma^* h_i$ in $\mathcal{H}_{B_2}$ or $\mathcal{H}_{B_1}$ are proportional, and one of them is nonzero. We have $\gamma^* \psi_i = a_i \otimes \gamma^* b_i + p_i \otimes q_i$, where p_i is the linear combination of c_i and g_i. We can find a state $q \in \mathcal{H}_{A_1}$ or $\mathcal{H}_{A_2}$ such that $q^* a_i = 0$ and $q^* g_i \neq 0$. So $\mathcal{R}(\alpha_{A_1 B_1})$ or $\mathcal{R}(\beta_{A_2 B_2})$ contains a product state $q^* p_i \otimes q_i$. It is in contradiction to the assumption. So we have proved the second assertion. This completes the proof. □

Next we generalize Lemma 3.2.13 (i) to the tensor product of many bipartite states.

Proposition 3.2.14 *Let* $\rho = \otimes_{j=1}^n \alpha_{A_j B_j}$ *be a bipartite state of systems* $A_1 \cdots A_n$:

$B_1 \cdots B_n$, where $\alpha_{A_jB_j}$ are bipartite states of the system A_jB_j, $j = 1, \cdots, n$, respectively. Suppose neither of $\mathcal{R}(\alpha_{A_jB_j})$ contains any product state. Then $\mathrm{SN}(\rho) > n$, and any decomposition of ρ consists of pure states of Schmidt rank at least $n+1$.

Proof. By the definition of Schmidt number, it suffices to prove the second assertion, that is any decomposition of ρ consists of pure states of Schmidt rank at least $n+1$. Suppose that is false. Let $\rho = \sum_i \xi_i \xi_i^*$ where

$$\xi_1 = a_{1A_1 \cdots A_n} \otimes b_{1B_1 \cdots B_n} + \cdots + a_{kA_1 \cdots A_n} \otimes b_{kB_1 \cdots B_n}, \tag{3.12}$$

is a bipartite pure state of Schmidt rank $k \le n$. Lemma 3.2.9 implies that there exists a nonzero state $\gamma \in \mathcal{H}_{A_1}$ such that the two states $\gamma^* a_1$ and $\gamma^* a_2$ in $\mathcal{H}_{A_2 \cdots A_n}$ are proportional, and one of them is nonzero. Let $\gamma' \in \mathcal{H}_{B_1}$ be a state such that $\eta := \gamma^*(\gamma'^* \xi_1) \ne 0$. So η is a bipartite pure state of Schmidt rank $k-1 \le n-1$. Next using Lemma 3.2.9 again, we can find a state $\delta \otimes \delta' \in \mathcal{H}_{A_2B_2}$ such that $\delta \otimes \delta'^* \eta \ne 0$ and has Schmidt rank at most $n-2$. Continuing in the same vein we can finally find a product state $\nu \in \mathcal{H}_{A_1 \cdots A_{n-1}:B_1 \cdots B_{n-1}}$ such that $\nu^* \psi_1 \in \mathcal{H}_{A_nB_n}$ is nonzero and has Schmidt rank at most one. So it is a product state in $\mathcal{R}(\rho_{A_nB_n})$. This is in contradiction to the assumption. So we have proved $\mathrm{SN}(\rho) > n$. This completes the proof. □

In Proposition 3.2.14, by choosing all $\alpha_{A_jB_j}$ to be PPT entangled states [ChDj13], one obtains a PPT entangled state ρ of Schmidt number n, where n can be greater than any prescribed integer. The state has equal birank (r, r) for some integer r. Moreover, we can obtain a PPT entangled state of an arbitrary Schmidt number by the upcoming Lemma 3.2.16 from the aforementioned state.

3.2.2 Approximation Problem of Schmidt Numbers

Different quantum states may play the same role in quantum-information tasks. Their similarity decides how they play in the tasks. The similarity of quantum states can be characterized by many quantum-information quantities, such as the fidelity, entanglement measure and equivalence under LOCC. In this subsection, we investigate the Schmidt number of projected states under different local projections. First of all we present the following definitions.

Definition 3.2.4 *Let ρ be an $M \times N$ entangled state, $k \in [1, M-1]$ an integer and*

P be a projection in $M_M(\mathbb{C})$. We define two quantities:

$$\mathrm{SN}_{\max}(\rho,k) := \max_P\{\mathrm{SN}(\sigma), \sigma=(P\otimes I_B)\rho(P^*\otimes I_B), \dim\ker(P)=k\}; \tag{3.13}$$
$$\mathrm{SN}_{\min}(\rho,k) := \min_P\{\mathrm{SN}(\sigma), \sigma=(P\otimes I_B)\rho(P^*\otimes I_B), \dim\ker(P)=k\}. \tag{3.14}$$

The two quantities in Definition 3.2.4 can be estimated in a few special cases. If $k=M-1$ then σ is separable. We have $\mathrm{SN}_{\max}(\rho,M-1)=\mathrm{SN}_{\min}(\rho,M-1)=1$. If $k=M-2$ then we have $\mathrm{SN}_{\max}(\rho,M-2),\mathrm{SN}_{\min}(\rho,M-2)\in[1,2]$. One may similarly prove that $\mathrm{SN}_{\max}(\rho,1),\mathrm{SN}_{\min}(\rho,1)\in[\mathrm{SN}(\rho)-1,\mathrm{SN}(\rho)]$. Lemma 3.2.10 (i) implies that

$$\begin{aligned}\max\{1,\mathrm{SN}(\rho)-k\}\le \mathrm{SN}_{\min}(\rho,k)\le \mathrm{SN}_{\max}(\rho,k)\\ \le \min\{\mathrm{SN}(\rho),M-k\}.\end{aligned} \tag{3.15}$$

The condition by which $1=\mathrm{SN}_{\min}(\rho,k)$ or $\mathrm{SN}(\rho)-k=\mathrm{SN}_{\min}(\rho,k)$ holds is in Lemma 3.2.10 (ii). If $\mathrm{SN}_{\max}(\rho,k)=\mathrm{SN}(\rho)$ for some k, then the space consisting all projected σ best approximates ρ in terms of Schmidt number. It is difficult in general to determine whether such a best approximation exists for an abitrary ρ. The equalities depend on the dimensions (M,N) as well as the pair $(\mathrm{SN}(\rho),k)$. To illustrate, let $k=1$ and pick ρ from the set of all 3×3 PPT states. By Corollary 3.2.11 we know that $\mathrm{SN}(\rho)=2$. Hence $1=\mathrm{SN}_{\max}(\rho,1)<\mathrm{SN}(\rho)=2$ since every 2×3 PPT states are separable. Consider ρ from the set of all 3×3 NPT states, then either $\mathrm{SN}(\rho)=2$ or $\mathrm{SN}(\rho)=3$. If $\mathrm{SN}(\rho)=3$, by Corollary 3.2.11, the projected states are NPT entangled states. Thus we have $2=\mathrm{SN}_{\max}(\rho,1)<\mathrm{SN}(\rho)=3$. If $\mathrm{SN}(\rho)=2$, consider the antisymmetric state $\rho=\sum_{j,k=1,j<k}^3(e_j\otimes f_k-e_k\otimes f_j)(e_j\otimes f_k-e_k\otimes f_j)^*$. Choose a projection $P=e_1e_1^*+e_2e_2^*$. Then $(P\otimes I_2)\rho(P^*\otimes I_2)$ is entangled. The next Lemma shows the relation between the Schmidt number of a quantum state and its projection in terms of Definition 3.2.4.

Lemma 3.2.15 $\mathrm{SN}_{\max}(\rho,k)=\mathrm{SN}(\rho)$ *holds for some k if and only if* $\mathrm{SN}_{\max}(\rho,1)=\mathrm{SN}(\rho)$.

Proof. The "if" part is trivial. It suffices to prove the "only if" part. Suppose $\mathrm{SN}_{\max}(\rho,k)=\mathrm{SN}(\rho)$. Since the Schmidt number does not increase under LOCC, we have

$$\mathrm{SN}_{\max}(\rho,k)\le\cdots\le\mathrm{SN}_{\max}(\rho,1)\le\mathrm{SN}(\rho). \tag{3.16}$$

So the assertion holds. This completes the proof. □

Note that $\mathrm{SN}_{\max}(\rho, k)$ may not equal $\max_Q\{\mathrm{SN}(\sigma), \sigma = (I_A \otimes Q)\rho(I_A \otimes Q^*), \dim\ker(Q) = k\}$. An example is $\rho = \xi\xi^* + (e_1 \otimes e_4)(e_1 \otimes e_4)^*$, and $\xi = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$, $k = 1$, $M = 3$ and $N = 4$. One can show that $\mathrm{SN}_{\max}(\rho, 1) = 2$ and $\max_Q\{\mathrm{SN}(\sigma), \sigma = (I_A \otimes Q)\rho(I_A \otimes Q^*), \dim\ker(Q) = 1\} = 3$. In general, we have the following Lemma.

Lemma 3.2.16 *Let ρ be an $M \times N$ entangled state, P and Q two nonzero projections respectively on $\mathcal{H}_A$ and $\mathcal{H}_B$. Then*
(i) the following three integer sets are the same,

$$\begin{aligned} & \{\mathrm{SN}(\sigma) : \sigma = (P \otimes I)\rho(P^* \otimes I),\ \forall P \neq 0\} \\ = & \{\mathrm{SN}(\sigma) : \sigma = (I \otimes Q)\rho(I \otimes Q^*),\ \forall Q \neq 0\} \\ = & \{1, 2, ..., \mathrm{SN}(\rho)\}. \end{aligned} \tag{3.17}$$

(ii) For any P there exists a Q such that

$$\mathrm{SN}\Big((P \otimes I)\rho(P^* \otimes I)\Big) = \mathrm{SN}\Big((I \otimes Q)\rho(I \otimes Q^*)\Big). \tag{3.18}$$

Proof. (i) Consider the set $A_k = \{\mathrm{SN}(\sigma) : \sigma = (P \otimes I)\rho(P^* \otimes I), \dim\ker P \leq k\}$. By Lemma 3.2.10 (i), we obtain $A_1 = \{\mathrm{SN}(\rho) - 1, \mathrm{SN}(\rho)\}$ or $A_1 = \{\mathrm{SN}(\rho)\}$. Denote by P_k a projection with $\dim\ker P_k = k$. Since any projection P_k can be written into $P_k = P_1 P_{k-1}$, we have $A_k = \{\mathrm{SN}(\sigma_k) : \sigma_k = (P_1 \otimes I)\sigma_{k-1}(P_1^* \otimes I), \sigma_{k-1} \in A_{k-1}\}$. Hence the set difference $A_k \backslash A_{k-1}$ is either an empty set or a set of single number by Lemma 3.2.10 (i). Using induction one has $A_{M-1} = \{1, ..., \mathrm{SN}(\rho)\}$. Similarly, we have the set $B_k = \{\mathrm{SN}(\sigma) : \sigma = (I \otimes Q)\rho(I \otimes Q^*), \dim\ker Q \leq k\}$ and $B_{N-1} = \{1, ..., \mathrm{SN}(\rho)\} = A_{M-1}$.
(ii) is an immediate consequence of (i). □

We also conjecture that for $k = 1, .., M-1$, the integer set $\{\mathrm{SN}(\sigma) : \sigma = (P \otimes I_B)\rho(P \otimes I_B), \dim\ker(P) = k\}$ is exactly the set of consecutive integers $\{\mathrm{SN}_{\min}(\rho, k), ..., \mathrm{SN}_{\max}(\rho, k)\}$. The conjecture holds when $k = M - 1, M - 2$ and 1, as shown by the argument below (3.14).

From Proposition 3.2.14 and Lemma 3.2.16, we obtain a main result in this chapter.

Theorem 3.2.17 *Given any positive integer r, there exist positive integers M, N and a bipartite PPT entangled state $\rho \in M_M(\mathbb{C}) \otimes M_N(\mathbb{C})$ of Schmidt number r.*

Proof. Using PPT entangled state $\alpha_{A_jB_j}$ in the process of construction in Proposition 3.2.14, we can obtain a PPT state ρ of arbitrarily large Schmidt number. By choosing proper projections P in Lemma 3.2.16, the projected state $(P \otimes I)\rho(P^* \otimes I)$ can attain any prescribed Schmidt number. Note that the projected state is also a PPT state and it completes the proof. $\square$

By the dual relation between quantum states and positive maps, Theorem 3.2.17 can be translated into the statement below.

Theorem 3.2.18 *Give any positive integer r, there exist positive integers M, N and an indecomposable map $\phi \in B(M_M(\mathbb{C}), M_N(\mathbb{C}))$ which is r-positive but not $(r+1)$-positive.*

Note that the dimensions of the underlying spaces where the map ϕ resides might be much bigger than r. For example, if one choose $\rho = \alpha_{A_1B_1} \otimes \alpha_{A_2B_2}$ where $\alpha_{A_1B_1}$ and $\alpha_{A_2B_2}$ are rank four 3×3 PPT entangled states, then ρ is a 9×9 PPT entangled state with $\text{SN}(\rho) \geq 3$. By Lemma 3.2.16 one can further project ρ to a $M \times N$ Schmidt number 3 state σ with $4 \leq \max\{M, N\} \leq 9$ if necessary. By (1.1) the state σ corresponds to an indecomposable 2-positive but not 3-positive map ϕ from $M_m(\mathbb{C})$ to $M_n(\mathbb{C})$, where $4 \leq \max\{m, n\} \leq 9$.

3.3 Schmidt Number of Multipartite States

We will use the physics notation to introduce the notion of Schmidt number of multipartite states in order to keep in consistent with the Definition 1.2.7 in the bipartite scenario. In this section we switch back to the physics notation for convenience. We shall denote by $\{|i\rangle_A : i = 0, \ldots, M-1\}$ an ONB of the subsystem $\mathcal{H}_A$.

Multipartite quantum states have a more complicated structure than that of bipartite states and have been extensively investigated in past years. For example the well-known n-partite Greenberger-Horne-Zeilinger (GHZ) state $\frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$ is the generalization of Bell state. It has been realized in experiments for small n with a high fidelity and play an important role in quantum computing. In this section we generalize the notion of Schmidt number to multipartite states. The *tensor rank* of an N-partite quantum state $|\psi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ of systems $A_1, \cdots, A_n$ is defined as the minimum integer r such that there exist r product states $|a_{j,1}, \cdots, a_{j,N}\rangle$ and $|\psi\rangle = \sum_{j=1}^r |a_{j,1}, \cdots, a_{j,N}\rangle$. For example the n-partite GHZ state has tensor rank two. Note that the tensor rank degenerate to the Schmidt rank for pure bipartite state.

So we still denote by $\text{rank}(|\psi\rangle)$ the tensor rank of $|\psi\rangle$. Now Definition 1.2.7 can be generalized to multipartite states as follows.

Definition 3.3.1 *A multipartite density matrix ρ has Schmidt number k if (i) for any decomposition $\{p_i > 0, |\psi_i\rangle\}$ of ρ, at least one of the vectors $|\psi_i\rangle$ has tensor rank at least k and (ii) there exists a decomposition of ρ with all vectors $|\psi_i\rangle$ of tensor rank at most k.*

For example, the three-qubit mixed state $\rho = |\alpha\rangle\langle\alpha| + |000\rangle\langle 000|$ where $|\alpha\rangle = |000\rangle + |111\rangle$ has Schmidt number two. To understand this fact, we assume that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ as an arbitrary decomposition of ρ. Using Lemma 3.2.8, one can obtain that there is always some $|\psi_i\rangle$ of tensor rank two. Then Definition 3.3.1 shows that $\text{SN}(\rho) = 2$, and that the Schmidt number of multipartite states does not increase under LOCC. So the Schmidt number is also an entanglement measure for multipartite states. Evidently, Definition 3.3.1 reduces to Definition 1.2.7 for bipartite states ρ. For simplicity we will regard tensor rank and Schmidt number as the same notion and use only Schmidt number. Further, the Schmidt number for bipartite and multipartite states are both invariant under ILOs. It is known that the Schmidt number is monotone decreasing under the local operations and classical communications [TeHo00]. So the Schmidt number is an entanglement monotone. Hence, the exact transformation under LOCC from a bipartite state $|\psi\rangle$ of smaller Schmidt rank to $|\varphi\rangle$ of bigger Schmidt rank is impossible. On the other hand, the transformation may be asymptotically realized by distilling EPR pairs from $|\psi\rangle$ and then preparing $|\varphi\rangle$. Third, it is known that for bipartite pure states $|\varphi\rangle$ we have $\text{SN}(|\varphi\rangle^{\otimes n}) = n\,\text{SN}(|\varphi\rangle)$. For multipartite pure states $|\psi\rangle$, we have $\text{SN}(|\psi\rangle^{\otimes n}) \le n\,\text{SN}(|\psi\rangle)$ and the inequality is strict for the multiqubit W state $|\psi\rangle \triangleq \frac{1}{\sqrt{k}}(|1\underbrace{0,...,0}_{k-1}\rangle + |01\underbrace{0,...,0}_{k-2}\rangle + \cdots + |\underbrace{0,...,0}_{k-1}1\rangle)$ and integers $n > 1$ [CCMV11]. Similar quantum measure of multipartite states has been mentioned in [EiBr01]. For a multipartite state ρ, the relation between the Schmidt number $\text{SN}(\rho)$ and the quantum measure $P(\rho)$ introduced in [EiBr01] is as follows. For any decomposition $\{\lambda_i, |\psi_i\rangle\}$ of a multipartite state ρ, we have the following inequality.

$$\begin{aligned}
&\sum_i \lambda_i \log_2 \text{rank}(|\psi_i\rangle) \\
&\le \log_2 \Big(\sum_i \lambda_i \,\text{rank}(|\psi_i\rangle)\Big) \\
&\le \log_2 \Big(\sum_i \lambda_i \max_i\{\text{rank}(|\psi_i\rangle)\}\Big) \\
&= \log_2 \max_i\{\text{rank}(|\psi_i\rangle)\}.
\end{aligned} \tag{3.19}$$

Here the first inequality in (3.19) follows from the concavity of logarithm. By minimizing (3.19) through all possible decompositions $\{\lambda_i, |\psi_i\rangle\}$ of a multipartite state ρ, one obtains the relation.

$$\begin{aligned} P(\rho) &= \min_{\{\lambda_i,|\psi_i\rangle\}} \sum_i \lambda_i \log_2(\mathrm{rank}(|\psi_i\rangle)) \\ &\leq \min_{\{\lambda_i,|\psi_i\rangle\}} \log_2 \max_i\{\mathrm{rank}(|\psi_i\rangle)\} \\ &= \log_2 \min_{\{\lambda_i,|\psi_i\rangle\}} \max_i\{\mathrm{rank}(|\psi_i\rangle)\} \\ &= \log_2 \mathrm{SN}(\rho). \end{aligned} \tag{3.20}$$

In the following subsections we construct and investigate three quantities of multipartite states, namely the expansion, coarse graining and joint Schmidt number. Their definitions are respectively given in Definition 3.3.2, 3.3.3 and 3.3.5. The expansion describes the global states whose reduced density operators are the target multipartite states. The coarse graining constructs multipartite states from the known ones by combining systems. The joint Schmidt number is another Schmidt number of multipartite states and different from Definition 3.3.1. The main results are given in Theorem 3.3.1, Lemma 3.3.2, Theorem 3.3.3 and Lemma 3.3.4. These establish the connection between the Schmidt number, local ranks of reduced density operators and global multipartite states.

3.3.1 Expansion

In this subsection we investigate the Schmidt number of multipartite states and their reduced density operators. We review the notion of expansion which works for the well-known quantum marginal problem.

Definition 3.3.2 *If ρ_A and ρ_B are the reduced density operators of a quantum state ρ_{AB}, then we say that ρ_{AB} is an expansion of ρ_A and ρ_B.*

An expansion of a quantum state describes the global physical environment when the quantum state is regarded as a local state. When ρ_{AB} is a pure state, it is also called the purification of ρ_A and ρ_B in literatures. For example if $\rho_A = \rho_B = \frac{1}{2}I_2$ then any two-qubit maximally entangled state ρ_{AB} is the expansion of ρ_A and ρ_B. Some ρ_A and ρ_B do not have any purification (or even expansion). Using the definition we have

Theorem 3.3.1 *(i) The Schmidt number of ρ_{ABC} is not smaller than the Schmidt number of ρ_{AB}, ρ_{AC} and ρ_{BC}.*
(ii) ρ_{AB} has Schmidt number at most k if and only if there is a tripartite state ρ_{ABC} of Schmidt number at most k.
(iii) Suppose $|\psi\rangle_{ABC}$ is the purification of ρ_{AB}. Then

$$\begin{aligned}
& \min\{\mathrm{SN}(\rho_{AB})\cdot\operatorname{rank}\rho_{AB},\ \ \operatorname{rank}\rho_A\cdot\operatorname{rank}\rho_B\} \\
\geq\ & \mathrm{SN}(|\psi\rangle_{ABC}) \\
\geq\ & \max\{\operatorname{rank}\rho_{AB},\ \ \operatorname{rank}\rho_A,\ \ \operatorname{rank}\rho_B\} \\
\geq\ & \mathrm{SN}(\rho_{AB}).
\end{aligned} \tag{3.21}$$

(iv) If ρ_{AB} is a PPT state, then the first two equalities in (3.21) *hold simultaneously if and only if* $\operatorname{rank}\rho_A\cdot\operatorname{rank}\rho_B=\operatorname{rank}\rho_{AB}$ *or* $\mathrm{SN}(\rho_{AB})=1$*, i.e. ρ_{AB} is a separable state.*
(v) If ρ_{AB} is a PPT state then the three equalities in (3.21) *hold simultaneously if and only if* $\operatorname{rank}\rho_A=\operatorname{rank}\rho_B=1$.
(vi) If $\rho_{AB}=|\psi\rangle\langle\psi|_{A_1B_1}\otimes\sum_i|ii\rangle\langle ii|_{A_2B_2}$ is a bipartite NPT state where $|\psi\rangle=\sum_j|jj\rangle$, $A=A_1A_2$, $B=B_1B_2$, then the last equality in (3.21) *holds. If ρ_{AB} has rank one then all three equalities in* (3.21) *hold.*

Proof. (i) Let $\rho_{ABC}=\sum_i|\psi_i\rangle\langle\psi_i|$ where each $|\psi_i\rangle$ has Schmidt number at most $k:=\mathrm{SN}(\rho_{ABC})$. So the pure states $\langle i|\psi_j\rangle$ has Schmidt number at most k. Since $\rho_{AB}=\mathrm{Tr}_C\,\rho_{ABC}=\sum_j\langle j|_C|\psi_i\rangle\langle\psi_i|j\rangle_C$, the assertion on ρ_{AB} holds. The other assertions can be proved similarly.

(ii) The "if" part follows from (i). To prove the "only if" part, suppose $\rho_{AB}=\sum_j|\psi_j\rangle\langle\psi_j|_{AB}$ where each $|\psi_j\rangle$ has Schmidt number at most k. Then $\rho_{ABC}=\sum_j|\psi_j\rangle\langle\psi_j|_{AB}\otimes|j\rangle\langle j|_C$ is an expansion of ρ_{AB} and has Schmidt number at most k.

(iii) Suppose $\rho_{AB}=\sum_{j=1}^{l}|\alpha_j\rangle\langle\alpha_j|_{AB}$ satisfies that $\mathrm{SN}(\alpha_j)\leq\mathrm{SN}(\rho_{AB})$. Without loss of generality, we may assume that the first $r:=\operatorname{rank}\rho_{AB}$ states $|\alpha_1\rangle,\cdots,|\alpha_r\rangle$ are linearly independent, and any $|\alpha_j\rangle$ is in the span of them. It is known that $|\psi\rangle_{ABC}=\sum_{j=1}^{l}|\alpha_j,u_j\rangle$ where the $|u_j\rangle$'s form a set of o. n. basis in $\mathbb{C}^l$ [**?**, Eq. (9.66)]. Hence

$$\mathrm{SN}(|\psi\rangle_{ABC})\leq\sum_{j=1}^{r}\mathrm{SN}(\alpha_j)\leq r\cdot\mathrm{SN}(\rho_{AB}). \tag{3.22}$$

Next the inequality $\operatorname{rank}\rho_A \operatorname{rank}\rho_B \geq k := \mathrm{SN}(|\psi\rangle_{ABC})$ follows from the definition of tensor rank. So we have proved the first inequality in (3.21). Let $\rho_{AB} = \sum_{i=1}^{r} |\alpha_i\rangle\langle\alpha_i|$ such that the $|\alpha_i\rangle$ are linearly independent. Then $|\psi\rangle_{ABC} = \sum_{i=1}^{r} |\alpha_i, i\rangle$, and thus $k \geq r$. Next the assertion $\mathrm{SN}(|\psi\rangle_{ABC}) \geq \max\{\operatorname{rank}\rho_A, \operatorname{rank}\rho_B\}$ follows by writing $|\psi\rangle_{ABC}$ as the bipartite state of systems $A : BC$ and $B : AC$. So we have proved the second inequality in (3.21). To prove the third inequality $\operatorname{rank}\rho_A \geq \mathrm{SN}(\rho_{AB})$ in (3.21), we notice that $\rho_{AB} = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ where each bipartite pure state $|\psi_i\rangle$ is an $M \times N$ state where $M \leq \operatorname{rank}\rho_A$ and $N \leq \operatorname{rank}\rho_B$. So the inequality holds.

(iv) The "if" part can be verified straightforwardly. Next we prove the “only if” part. Since ρ_{AB} is a PPT state, then $\operatorname{rank}\rho_{AB} \geq \max\{\operatorname{rank}\rho_A,\ \operatorname{rank}\rho_B\}$ [HSTT03]. Hence the assumption of the “only if” part is equivalent to

$$\begin{aligned} & \min\{\mathrm{SN}(\rho_{AB}) \cdot \operatorname{rank}\rho_{AB},\ \operatorname{rank}\rho_A \cdot \operatorname{rank}\rho_B\} \\ = \quad & \mathrm{SN}(|\psi\rangle_{ABC}) \\ = \quad & \operatorname{rank}\rho_{AB}. \end{aligned} \tag{3.23}$$

If $\min\{\mathrm{SN}(\rho_{AB}) \cdot \operatorname{rank}\rho_{AB},\ \operatorname{rank}\rho_A \cdot \operatorname{rank}\rho_B\} = \mathrm{SN}(\rho_{AB}) \cdot \operatorname{rank}\rho_{AB}$ then one obtains $(\mathrm{SN}(\rho_{AB}) - 1) \cdot \operatorname{rank}(\rho_{AB}) = 0$. Hence ρ_{AB} is separable. On the other hand if $\min\{\mathrm{SN}(\rho_{AB}) \cdot \operatorname{rank}\rho_{AB},\ \operatorname{rank}\rho_A \cdot \operatorname{rank}\rho_B\} = \operatorname{rank}\rho_A \cdot \operatorname{rank}\rho_B$ then it is obvious that $\operatorname{rank}\rho_A \cdot \operatorname{rank}\rho_B = \operatorname{rank}\rho_{AB}$.

(v) The assertion follows from (iv), (3.21) and $\operatorname{rank}\rho_{AB} \geq \max\{\operatorname{rank}\rho_A,\ \operatorname{rank}\rho_B\}$.

(vi) The assertion can be verified straightforwardly using Lemma 3.2.2, because the states $|\psi\rangle_{A_1B_1} \otimes |jj\rangle_{A_2B_2}$ are orthogonal each other. This completes the proof. □

Remarks:

1. When $k = 2$, assertion (ii) gives a necessary and sufficient condition for whether ρ has Schmidt number at most two. Besides the equality $\mathrm{SN}(\rho_{ABC}) = \mathrm{SN}(\rho_{AB}) = \mathrm{SN}(\rho_{BC}) = \mathrm{SN}(\rho_{AC})$ may hold for some ρ_{ABC}. An example is the three-qubit state $|000\rangle + |a, a, a\rangle$ where $|a\rangle = |0\rangle + |1\rangle$.

2. It is possible that

$$\begin{aligned}
\mathrm{SN}(\rho_{A_1\cdots A_n}) &> \sum_{1\le j_1<j_2\le n} \mathrm{SN}(\rho_{A_{j_1}A_{j_2}}) \\
&+ \sum_{1\le j_1<j_2<j_3\le n} \mathrm{SN}(\rho_{A_{j_1}A_{j_2}A_{j_3}}) \\
&+ \cdots \\
&+ \sum_{1\le j_1<\cdots<j_{n-1}\le n} \mathrm{SN}(\rho_{A_{j_1}\cdots A_{j_{n-1}}}). \qquad (3.24)
\end{aligned}$$

For example, the inequality holds when ρ is the d-level GHZ state $\sum_{j=1}^{d} |jj\cdots j\rangle$ when d is sufficiently big. The reason is that any k-partite reduced density operator σ of ρ is a separable state, i.e., $\sigma = \sum_j p_j |a_{j,1},\cdots,a_{j,k}\rangle\langle a_{j,1},\cdots,a_{j,k}|$. Hence $\mathrm{SN}(\sigma)=1$. All together we have $\sum_{k=2}^{n-1}\binom{n}{k} = 2^n - \binom{n}{1} - \binom{n}{n} - \binom{n}{0} = 2^n - n - 2$ number of terms. If each system has dimension $d_k > 2^n - n - 2$, then any d level GHZ state with $d > 2^n - n - 2$ will satisfy the inequality. Since the Schmidt number is a multipartite entanglement measure, (3.24) shows the monogamy relation for some states.

3. In assertion (iii), we have shown the relation between the Schmidt number, the rank and the purification of a bipartite state. The known inequality $\operatorname{rank}\rho_A \cdot \operatorname{rank}\rho_B \ge \operatorname{rank}\rho_{AB}$ holds for any state ρ_{AB}. Eq. (3.21) gives the inequality $\operatorname{rank}\rho_A \cdot \operatorname{rank}\rho_B \ge \mathrm{SN}(|\psi\rangle_{ABC}) \ge \operatorname{rank}\rho_{AB}$ which is stronger than the known inequality.

4. In assertion (iv), if the state ρ_{AB} is not PPT then it may still make the first two equalities in (3.21) hold. For example ρ_{AB} is the bipartite pure entangled state. A more complicated example is the mixed entangled state $\rho_{AB} = |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta|$ where $|\alpha\rangle = |11\rangle + |22\rangle$ and $|\beta\rangle = |33\rangle + |44\rangle$. One can verify that the first two equalities in (3.21) holds since $\operatorname{rank}\rho_A = \operatorname{rank}\rho_B = 4$, $\mathrm{SN}(\rho_{AB}) = \operatorname{rank}\rho_{AB} = 2$ and $\mathrm{SN}(|\psi\rangle_{ABC}) = 4$. On the other hand, the second equality in (3.21) fails when $|\alpha\rangle = |01\rangle + |10\rangle$ and $|\beta\rangle = |00\rangle$. One can show that $\mathrm{SN}(|\psi\rangle_{ABC}) = 3 > \mathrm{SN}(\rho_{AB}) = \operatorname{rank}\rho_{AB} = 2$. It is an interesting question to investigate when the last equality in (3.21) holds.

5. For any tripartite state $|\psi\rangle_{ABC}$, if we regard it as a bipartite state over the split of systems A and BC, then we obtain $\operatorname{rank}\rho_A = \operatorname{rank}\rho_{BC}$. Similarly one obtains $\operatorname{rank}\rho_B = \operatorname{rank}\rho_{AC}$, and $\operatorname{rank}\rho_C = \operatorname{rank}\rho_{AB}$. So only three of the six parameters $\operatorname{rank}\rho_A, \operatorname{rank}\rho_B, \operatorname{rank}\rho_C, \operatorname{rank}_{AB}, \operatorname{rank}_{AC}, \operatorname{rank}_{BC}$ are independent. In fact we have chosen the three parameters $\operatorname{rank}\rho_A, \operatorname{rank}\rho_B$ and $\operatorname{rank}\rho_{AB}$ in (3.21). The other two parameters $\mathrm{SN}(\rho_{AB})$ and $\mathrm{SN}(|\psi\rangle_{ABC})$ are also independent from the three parameters. On the other hand the six parameters of a mixed tripartite state may be independent from each other, and the investigation is more complicated. For readers' reference,

the relation between the ranks of global and local systems for the entropy has been recently investigated [CHLW14].

3.3.2 Coarse Graining

In this subsection we investigate the Schmidt number of multipartite states in terms of its coarse graining. The latter is defined as follows.

Definition 3.3.3 *(i) Let ρ be an n-partite quantum state in the systems $A_1, \cdots, A_n$. If we partition the systems into m disjoint parties $B_1, \cdots, B_m$ then we obtain a new m-partite quantum state σ. We call σ a coarse graining of ρ.*
(ii) The multipartite PPT states w.r.t. the systems $A_1, A_2, \cdots, A_n$ are defined as the states any bipartition of whom is a PPT state. We denote by ρ^{Γ_j} the partial transpose w.r.t. system A_j.

For example if $|\psi\rangle = |000\rangle + |111\rangle$, $B_1 = A_1$, and $B_2 = A_2A_3$, then $|\varphi\rangle = |\psi\rangle = |00\rangle + |13\rangle$ where $|0\rangle_{B_2} = |00\rangle_{A_2A_3}$ and $|3\rangle_{B_2} = |11\rangle_{A_2A_3}$.

We explain the coarse graining from the point of view of quantum information. In a multipartite state $|\psi\rangle$, some of the n systems can be combined so that they perform collective operation, and create more quantum correlation quantitatively and qualitatively in $|\psi\rangle$. So the coarse graining of $|\psi\rangle$ represent different entanglement structure from $|\psi\rangle$. The coarse graining has been used to investigate the geometric measure of entanglement [ZCH10].

Lemma 3.3.2 *(i) The Schmidt number of a multipartite pure state is not smaller than that of its coarse graining.*
(ii) The multipartite state ρ w.r.t. the systems $A_1, A_2, \cdots, A_n$ and its partial transpose ρ^{Γ_j} w.r.t system A_j for any j are simultaneously separable or not.

Proof. (i) Since the tensor rank is non-increasing if subsystems are grouped together, hence the assertion holds.
(ii) If ρ is separable, then its partial transpose ρ^{Γ_j} is separable by definition. □

3.3.3 Joint Schmidt Numbers

In this subsection we construct another version of Schmidt number of multipartite states which is different from Definition 3.3.1. We begin by reviewing the version of pure multipartite states constructed in [HaKy16jmp].

Definition 3.3.4 *If the multipartite state* $|\phi\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ *has Schmidt number* s_l *under the bi-partition* $\mathcal{H}_l \otimes (\otimes_{j\neq l}\mathcal{H}_j)$*, then we say that* $|\phi\rangle$ *has joint Schmidt number* $\mathrm{JSN}(\phi) = (s_1, ..., s_n)$*.*

For example, the genuinely entangled multiqubit state has joint Schmidt number $(2, \cdots, 2)$. Essentially, the definition arises in the different bi-partitions of the systems. Given two n-partite states ρ and σ with $\mathrm{JSN}(\rho) = (s_1, ..., s_n)$ and $\mathrm{JSN}(\sigma) = (t_1, ..., t_n)$, we say that σ dominates ρ and denote it by $\mathrm{JSN}(\rho) \leq \mathrm{JSN}(\sigma)$ if $s_i \leq t_i$ for $i = 1, ..., n$. So two tuples $(s_1, \cdots, s_n)$ and $(t_1, \cdots, t_n)$ are equal when they dominate each other.

Definition 3.3.5 *The multipartite state* ρ *in the system* $\prod_{i=1}^n A_i$ *has joint Schmidt number* $(s_1, \ldots, s_n)$ *if it has Schmidt number* s_l *under the system bipartition of* $A_l : \prod_{i\neq l} A_i$*. If in addition there exists a decomposition* $\rho = \sum_i |\phi_i\rangle\langle\phi_i|$ *with all* $\mathrm{JSN}(|\phi_i\rangle) \leq (s_1, \ldots, s_n)$*, then we say the decomposition is a balanced decomposition.*

For example, the three-qubit state $|\psi\rangle\langle\psi| + |000\rangle\langle000|$ has joint Schmidt number $(2, 2, 2)$ where $|\psi\rangle = |001\rangle + |010\rangle + |100\rangle$. The definition implies that a multipartite state is separable if and only if it has a balanced decomposition with joint Schmidt number $(1, \ldots, 1)$. Furthermore, for any local operators $V = \otimes_{j=1}^n V_j$, one can show that $\mathrm{JSN}(V\rho V^*) \leq \mathrm{JSN}(\rho)$. Hence the joint Schmidt number is a multipartite entanglement monotone and is physically meaningful. This is similar to the role of Schmidt number for bipartite states. We further investigate the mathematical relation of them.

Theorem 3.3.3 *(i) Let* $|\psi\rangle$ *be a multipartite state of* $\mathrm{JSN}(|\psi\rangle) = (s_1, ..., s_n)$*. Then* $\max_{j=1,\ldots,n}\{s_j\} \leq \mathrm{SN}(|\psi\rangle) \leq \min_{j=1,\ldots,n}\{\frac{\Pi_{i=1}^n s_i}{s_j}\}$*.*
(ii) If $|\psi\rangle$ *is separable under* $(n-1)$ *many bi-partitions, then* $|\psi\rangle$ *is separable.*

Proof. (i) The lower bound $\max_{j=1,\ldots,n}\{s_j\} \leq \mathrm{SN}(\psi)$ follows from the definition of Schmidt number. We will prove the assertion that $\mathrm{SN}(\psi) \leq \prod_{i\neq n} s_i$ and one can similarly prove the assertion. By definition we have n ways of bipartition, namely $|\psi\rangle = \sum_{i=1}^{s_l} |a_i^l\rangle_{A_l} \otimes |b_i^l\rangle_{\prod_{j\neq l} A_j}$ where $|a_i^l\rangle$ are orthonormal states and the superscript $l \in \{1, ..., n\}$. Hence $|\psi\rangle = \sum_{i=1}^{s_l} |a_i^l\rangle\langle a_i^l|_{A_l}|\psi\rangle$. By using this equation for $l = 1, \ldots, n-1$

we have

$$\begin{aligned} |\psi\rangle &= \otimes_{l=1}^{n-1} \sum_{i=1}^{s_l} |a_i^l\rangle\langle a_i^l|_{A_i}|\psi\rangle \\ &= \sum_{i_1=1}^{s_1} \cdots \sum_{i_{n-1}=1}^{s_{n-1}} |a_{i_1}^1, \cdots, a_{i_{n-1}}^{n-1}\rangle_{A_1\cdots A_{n-1}} |\psi_{i_1,\cdots,i_{n-1}}\rangle, \end{aligned} \tag{3.25}$$

where $|\psi_{i_1,\ldots,i_{n-1}}\rangle = \langle a_{i_1}^1, \cdots, a_{i_{n-1}}^{n-1}|\psi\rangle$ is a vector in $\mathcal{H}_n$. So the assertion follows.

(ii) If $|\psi\rangle$ is separable under $(n-1)$ many bi-partitions, then $\text{JSN}(|\psi\rangle) = (1, 1, \ldots, 1)$. By the inequality in assertion (i), we deduce that $\text{SN}(|\psi\rangle) = 1$. □

Remarks:
1. The bound in Theorem 3.3.3 (i) is tighter than that in [HaKy16jmp, Theorem 4.2], which says $\text{SN}(\rho) \leq \prod_{i=1}^n s_i$. For example consider the tripartite state $|\psi\rangle = |111\rangle + |122\rangle + |213\rangle + |224\rangle$. One can verify that $\text{SN}(\psi) = 4$ and $\text{JSN}(|\psi\rangle) = (2, 2, 4)$. So $\text{SN}(\psi) = s_1 s_2 < s_1 s_2 s_3 = 16$. On the other hand, any 4-partite pure state $|\varphi\rangle_{A_1A_2A_3A_4}$ can be regarded as a tripartite state, say $|\alpha\rangle_{A_1,A_2,A_3A_4}$ in terms of Definition 3.3.3. If $\text{JSN}(|\varphi\rangle) = (s_1, s_2, s_3, s_4)$ then $\text{JSN}(|\alpha\rangle) = (s_1, s_2, s_3')$. So Lemma 3.3.2 says that $\text{SN}(\psi) \geq \text{SN}(\alpha)$, and Theorem 3.3.3 says that $s_1 s_2 \geq \text{SN}(\alpha)$. Hence

$$\min\{\text{SN}(\psi), s_1 s_2\} \geq \text{SN}(\alpha). \tag{3.26}$$

2. The condition of $(n-1)$ many bipartitions in Theorem 3.3.3 (ii) is necessary. Indeed a multipartite state $|\psi\rangle$ may be entangled if its $(n-2)$ many bipartitions are all separable. An example is the tripartite state $|\psi\rangle = |000\rangle + |110\rangle$.

3. In spite of Theorem 3.3.3 (ii), the biseparability via all bi-partitions does not imply the separability of multipartite mixed states. An example is the 3-qubit PPT entangled state $\rho = I - \sum_{j=1}^4 |a_i, b_i, c_i\rangle\langle a_i, b_i, c_i|$ where $\{|a_i, b_i, c_i\rangle\}$ is a 3-qubit UPB. One can show that $\text{JSN}(\rho) = (1, 1, 1)$, and ρ has Schmidt rank two. Since $\text{SN}(\rho) = 2 > 1^3/1 = 1$, Theorem 3.3.3 (i) cannot be generalized to mixed states.

In fact, any multipartite PPT state ρ of rank at most three, or any non-three-qubit and non-two-qutrit PPT state of rank four is separable [ChDj13]. Thus it has joint Schmidt number $(1, 1, \cdots, 1)$. On the other hand, it does not have a balanced decomposition, because it is entangled. One can verify that for any $j = 1, 2, 3$, ρ^{Γ_j} is still a PPT entangled state of rank at most four, and satisfies $\text{JSN}(\rho^{\Gamma_j}) = \text{JSN}(\rho) = (1, 1, 1)$ and

$\mathrm{SN}(\rho^{\Gamma_j}) = \mathrm{SN}(\rho) = 2$. For general entangled states we have the following results.

Lemma 3.3.4 *(a) Let ρ be a multipartite entangled PPT state of rank less than four. Then*
(i) ρ and its partial transpose w. r. t. any systems, when regarded as bipartite states, all have Schmidt number two.
(ii) If ρ is not a two-qutrit state then $\mathrm{JSN}(\rho) = (1, \cdots, 1)$*.*
(b) Any multipartite entangled PPT state with Schmidt number at least 3 when regarded as bipartite states, has rank at least 5.

Proof. (a)(i) All PPT entangled states with rank at most 3 are separable [ChDj13]. It is known that any entangled PPT state ρ of rank four is either a three-qubit or a two-qutrit state [ChDj13]. The assertion holds when ρ is a two-qutrit state by Corollary 3.2.11. On the other hand if ρ is a three-qubit state, then $\mathrm{JSN}(\rho) = (1, 1, 1)$ [ChDj13]. So ρ is the convex sum of product states over the bipartition of spaces $\mathcal{H}_1 : \mathcal{H}_{2,3}$. Hence the assertion also holds.

(ii) The assertion can be proved by the argument similar to that of (i).

(b) Immediate from (i).This completes the proof. □

Lemma 3.3.4 (iii) restricts the rank of desired states whose Schmidt number is different from that of its partial transpose. So far there is no example or proof for the existence of such states.

3.4 Related Problems

In this section we introduce some open problems on the Schmidt number. Let ρ be a bipartite state, P a projection on $\mathcal{H}_A$, and $P^\perp = 1 - P$ the projection orthogonal to P. Let $\alpha = (P \otimes I)\rho(P \otimes I)$ and $\beta = (P^\perp \otimes I)\rho(P^\perp \otimes I)$. Then we may expect that $\mathrm{SN}(\rho) \le \mathrm{SN}(\alpha) + \mathrm{SN}(\beta)$. However this inequality is generally incorrect and we give a counterexample. Let $\rho = |\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi| + |\omega\rangle\langle\omega|$ where $|\psi\rangle = |11\rangle + |22\rangle$, $|\varphi\rangle = |33\rangle + |44\rangle + |55\rangle$, and $|\omega\rangle = |33\rangle - |44\rangle + |66\rangle$. Let $P = |1\rangle\langle 1| + |3\rangle\langle 3| + |4\rangle\langle 4|$. One can verify that α and β are both separable states. We claim that $\mathrm{SN}(\rho) = 3$ and thus the inequality is wrong. To prove the claim, we note that the maximal Schmidt rank of any state in $\mathcal{R}(\rho)$ is three, then the claim follows from the definition of Schmidt number and Lemma 3.2.8.

Lemma 3.2.15 shows that if $\mathrm{SN}_{\min}(\rho,k)=\mathrm{SN}(\rho)$ or $\mathrm{SN}_{\max}(\rho,k)=\mathrm{SN}(\rho)$ for some k, then the minimum k is one. On the other hand $\mathrm{SN}_{\min}(\rho,k)=\mathrm{SN}_{\max}(\rho,k)=1$ when $k=M-1$.

Conjecture 3.4.1 *(i) There exists a PPT state ρ such that* $\mathrm{SN}(\rho)>\mathrm{SN}(\rho^\Gamma)$*.*
(ii) Such ρ exists in $M\times N$ system where $3\le M\le N$ and $MN\ge 12$. The simplest ρ is a 3×4 PPT state of BSN $(2,3)$.
(iii) If the simplest ρ in (ii) exists then $\mathrm{SN}(\rho^{\otimes 2})$ has BSN $(4,9)$.
(iv) If (i) holds then there exists ρ constructed from a UPB $\{|a_j,b_j\rangle\}$, i.e., $\rho=I-\sum_j|a_j,b_j\rangle\langle a_j,b_j|$.

Since Schmidt number is an entanglement measure, the equality $\mathrm{SN}(\rho)=\mathrm{SN}(\rho^\Gamma)$ would imply that ρ and ρ^Γ have the same entanglement. However, to find an example for Conjecture 3.4.1 (ii), one has to find a 3×4 entangled PPT state with Schmidt number 3. No concrete example has been given in the literature yet. The existence of a 3×4 PPT state ρ with $\mathrm{SN}(\rho)=3$ is equivalent to the existence of an indecomposable 2-positive map in $B(M_3(\mathbb{C}),M_4(\mathbb{C}))$. Note that if such a state exists, then it may provide a candidate for an example for Conjecture 3.4.1. One need to further check $\mathrm{SN}(\rho^\Gamma)=2$ besides $\mathrm{SN}(\rho)=3$. More generally, one may pose the following.

Conjecture 3.4.2 *For any positive integer L, there is a PPT state ρ such that* $|\,\mathrm{SN}(\rho)-\mathrm{SN}(\rho^\Gamma)|\ge L$*.*

Chapter 4

Distillability Problem

This chapter introduces some attempts of solving a special case of distillability Problem in an ongoing collaborative project with Prof. Chen and Prof. Tang. The general distillability problem is raised in [HHH98].

4.1 Background and Current Status

Let us introduce the distillability of quantum states. Denote by $\mathcal{H}_k$ the k-dimensional Hilbert space.

Definition 4.1.1 *Let ρ be an $m \times n$ bipartite state.*
(i) ρ is k-distillable if there exist rank two projections $P : \mathcal{H}_m^{\otimes k} \to \mathcal{H}_2$ and $Q : \mathcal{H}_n^{\otimes k} \to \mathcal{H}_2$ such that the projected state $(P \otimes Q)\rho^{\otimes k}(P^* \otimes Q^*)$ is an entangled state. Otherwise ρ is k-undistillable.
(ii) ρ is distillable if ρ is k-distillable for some integer k. If ρ is k-undistillable for any integer k then ρ is undistillable.

For example, any 3×3 NPT states are 1-distillable by Corollary 3.2.11. If ρ is a PPT state, so is $\rho^{\otimes k}$. Since the projected state $(P \otimes Q)\rho^{\otimes k}(P^* \otimes Q^*)$ is a 2×2 bipartite state, it is separable by the Peres-Horodecki criterion. Hence ρ is PPT implies ρ is undistillable [HHH98].

Moreover, a undistillable entangled state is called to process bound entanglement. Although PPT entangled states are undistillable, the well-known distillability problem conjectures that some undistillable entangled states are NPT, equivalently, there exists NPT bound entanglement. The general belief to the conjecture is "Yes" but there is no proof or counterexample yet. In 2000, important progress was made on the distilability problem [DSSTT00]. That is, all bipartite NPT states can be converted into $n \times n$ NPT Werner states $\rho(p,n)$ with some $n \geq 2$. When $n = 2$ every NPT state is trivially distillable. It suffices to consider the following unnormalized states with $n \geq 3$.

$$\rho(p,n) := I \otimes I - p \sum_{i,j=1}^{n} |i,j\rangle\langle j,i|, \tag{4.1}$$

for $p \in [-1,1]$. It is known that $\rho(p,n)$ is

1) separable when $p \in [-1, \frac{1}{n}]$;

2) NPT and 1-undistillable when $p \in (\frac{1}{n}, \frac{1}{2}]$;

3) NPT and 1-distillable when $p \in (\frac{1}{2}, 1]$.

We shall call $\rho(1/2,n)$ the critical Werner state. It can be converted into $\rho(p,n)$ with $p \in (0,1/2)$ under LOCC. So the non-distillability of the critical Werner state implies that of $\rho(p,n)$ with $p \in (0,1/2)$. It is conjectured that the critical Werner state is undistillable, and the conjecture is equivalent to the distillability problem. One can show that

$$\rho(p,n) = (1-p)S_p + (1+p)A_p \tag{4.2}$$

where $S_p = \sum_{i=1}^{n} |ii\rangle\langle ii| + \sum_{i,j=1,i>j}^{n} \frac{|ij\rangle+|ji\rangle}{\sqrt{2}} \frac{\langle ij|+\langle ji|}{\sqrt{2}}$ and $A_p = \sum_{i,j=1,i>j}^{n} \frac{|ij\rangle-|ji\rangle}{\sqrt{2}} \frac{\langle ij|-\langle ji|}{\sqrt{2}}$ are respectively the projections of symmetric and antisymmetric subspaces in $\mathcal{H}$. So $\rho(p,n)$ has Schmidt number at most two. In particular it has Schmidt number exactly two when $p \in (\frac{1}{n}, 1]$. Let ρ^{Γ_X} be the partial transpose of the bipartite state ρ w.r.t. the system $X = A, B$. We shall refer to symmetric density matrices as the states with symmetric matrices. For any symmetric density matrix ϱ, we have $\varrho^{\Gamma_A} = \varrho^{\Gamma_B}$. In particular the equality holds when ϱ is the Werner state. The following Lemma from [Wa04] establishes examples of n-undistillable but $(n+1)$-distillable states.

Lemma 4.1.1 *For any choice of integers $n \geq 3$ and $k \geq 1$, there exists an $n^2 \otimes n^2$ bipartite mixed quantum state that is distillable but k-undistillable.*

Let us recall the *reduction map* $\Lambda(\alpha) = (\operatorname{Tr}\alpha)I - \alpha$ for any positive semidefinite matrix α [HoHo99]. Let Λ_A and Λ_B be the maps respectively acting on the system A and B. One can show

$$\begin{aligned} \Lambda_A(\rho) &= I_A \otimes \rho_B - \rho, \\ \Lambda_B(\rho) &= \rho_A \otimes I_B - \rho, \end{aligned} \tag{4.3}$$

for any bipartite state ρ. The reduction map is positive and not 2-positive [To85, TeHo00]. If both matrices in (4.3) are positive semidefinite then we say that ρ satisfies the reduction criterion. Otherwise ρ violates the reduction criterion, i.e., one of the two matrices in (4.3) is not positive semidefinite. It is known that if the reduction criterion is violated then ρ is 1-distllable [HoHo99]. Since we can assume that $\rho_B = I_B/\dim(\mathcal{H}_B)$ up to ILOs, we obtain that all states with $\operatorname{rank}\rho < \max\{\operatorname{rank}\rho_A, \operatorname{rank}\rho_B\}$ is 1-distillable. This is also the main result of [HSTT03]. Note that all NPT states with $\operatorname{rank}\rho = \max\{\operatorname{rank}\rho_A, \operatorname{rank}\rho_B\}$ is also 1-distillable [ChDj11jpa]. Recently, it has been proved that all two-qutrit NPT states of rank four are also 1-distillable [ChDj16]. In the same paper, a family of 1-undistillable two-qutrit NPT states of rank five has also been constructed using edge PPT entangled states. It is conjectured that there exist undistillable two-qutrit NPT states of rank five [ChDj16].

Finally the reduction criterion is weaker than the PPT criterion. There are a few theoretical tools containing this relation in [HaCh11].

4.2 Some Attempts

Denote $|\Psi_n\rangle := \sum_{i=1}^n |ii\rangle$ as the maximally entangled state, then $|\Psi_n\rangle\langle\Psi_n| = \sum_{i,j=1}^n E_{ij}\otimes E_{ij}$. Define a completely positive map $\psi_n \in B(M_n(\mathbb{C}), M_n(\mathbb{C}))$ by the critical Werner state $\rho(1/2, n) = (id_n \otimes \psi_n)(|\Psi_n\rangle\langle\Psi_n|)$. Since the critical Werner state is NPT, ψ_n is not completely copositive. In the following we characterize $\tau_n \circ \psi_n$. Note that $\rho_w^{\Gamma_B}(1/2, n) = (id_n \otimes \tau_n)\rho(\frac{1}{2}) = (id_n \otimes \tau_n)(id_n \otimes \psi_n)(|\Psi_n\rangle\langle\Psi_n|) = (id_n \otimes (\tau_n \circ \psi_n))(|\Psi_n\rangle\langle\Psi_n|) = \sum_{i,j=1}^n E_{ij} \otimes (\tau_n \circ \psi_n(E_{ij}))$. By calculation we have $\rho_w^{\Gamma_B}(1/2, n) = \frac{1}{2}(2\sum_{i,j=1}^n E_{ii} \otimes E_{jj} - \sum_{i,j=1}^n E_{ij} \otimes E_{ij})$. Therefore we obtain the *Werner map*

$$\tau_n \circ \psi_n = \operatorname*{Tr}_n - id_n/2, \tag{4.4}$$

where $\operatorname*{Tr}_n$ is the trace map on $B(M_n(\mathbb{C}), M_n(\mathbb{C}))$ defined as $\operatorname*{Tr}_n(X) = trace(X)I_n$. It is completely copositive, and not completely positive by Theorem 1.1.3. In fact, $\tau_n \circ \psi_n$ is a 2-positive map by Tomiyama in [To85, Theorem 2]. The map $(n-1)\operatorname*{Tr}_n - id_n$ is an $(n-1)$-positive but not CP map [Ch72].

Further the map $(\tau_n \circ \psi_n)^{\otimes k}$ is completely copositive for any positive integer k. This is a corollary of the following general observation.

Lemma 4.2.1 *The tensor powers of any completely positive map are completely positive, and correspondingly the tensor powers of any completely copositive map are* *completely copositive.*

Proof. The assertion follows from the definition of complete positive and copositive maps. If ψ is completely positive, using Choi-Kraus decomposition for the finite dimension case, any tensor power still has an associated Choi-Kraus decomposition, hence completely positive. Correspondingly for the completely copositive case.

□

The lemma implies that $(\tau_n \circ \psi_n)^{\otimes k}$ is a decomposable map. Besides this two facts, we know few about other properties of the map $(\tau_n \circ \psi_n)^{\otimes k}$ even when $k = 2$. On the other hand, the tensor product of two positive maps may not be positive. For example the map $id_n + \tau_n$ is positive and $(id_n + \tau_n) \otimes (id_n + \tau_n)$ is not positive when acting on the input matrix $|\Psi_2\rangle\langle\Psi_2|$.

The following result is from Theorem 4 of [St16jmp].

Lemma 4.2.2 *Let $\phi \in B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ be a positive map such that $\phi^{\otimes k}$ is 2-positive for all positive integers k. Then the Choi matrix $[\phi(E_{ij})]_{i,j=1}^m$ is undistillable if and only if $(\tau_n \circ \phi)^{\otimes k}$ is 2-positive for all integers k.*

We denote by $\omega_{p,n}$ the completely positive map associated with the Werner state, i.e., $\rho(p, n) = (id_n \otimes \omega_{p,n})(|\Psi_n\rangle\langle\Psi_n|)$. In particular $\omega_{1/2,n} = \psi_n$. From (4.2) we have the Choi-Kraus decomposition of $\omega_{p,n}$ as follows.

$$\begin{aligned}
\omega_{p,n}(*) &:= \operatorname*{Tr}_n -p \cdot id_n(*) \\
&= \sum_{i=1}^n (1-p)|i\rangle\langle i|(*)|i\rangle\langle i| + \sum_{i,j=1, i>j}^n (1-p)\frac{|i\rangle\langle j| + |j\rangle\langle i|}{\sqrt{2}}(*)\frac{|i\rangle\langle j| + |j\rangle\langle i|}{\sqrt{2}} \\
&+ \sum_{i,j=1, i>j}^n (1+p)\frac{|i\rangle\langle j| - |j\rangle\langle i|}{\sqrt{2}}(*)\frac{|j\rangle\langle i| - |i\rangle\langle j|}{\sqrt{2}}. \qquad (4.5)
\end{aligned}$$

Hence one set of operators in the Choi-Kraus decomposition of $\omega_{p,n}$ is

$$\{\sqrt{1-p}|i\rangle\langle i|, \sqrt{\frac{1-p}{2}}(|i\rangle\langle j| + |j\rangle\langle i|), \sqrt{\frac{1-p}{2}}(|i\rangle\langle j| - |j\rangle\langle i|), i, j = 1, ..., n\ i \neq j\}.$$

The distillability problem can be reformulated as a problem of positive maps, i.e., the critical Werner state is k-undistillable if and only if the map $(\tau_n \circ \psi_n)^{\otimes k}$ is 2-positive [DSSTT00, Theorem 4]. Indeed, $\tau_n \circ \psi_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the first example of a $(n-1)$-positive but not n-positive (so completely positive) map [Ch72]. So we have obtained the known fact that the critical Werner state is 1-undistillable. Next, the 2-undistillability of the critical Werner state remains unknown and is equivalent to the following conjecture [DSSTT00, Theorem 4].

Conjecture 4.2.1 *The map* $(\tau_n \circ \psi_n)^{\otimes 2}$ *is* 2*-positive for* $n \geq 3$.

For $n = 3$, the associated positive map $\tau_3 \circ \psi_3$ is from $M_3(\mathbb{C})$ to $M_3(\mathbb{C})$. In this case, Conjecture 4.2.1 is supported by robust numerical test [ViDo06]. In case of high dimensions and multiple tensors, there is no support from the numerical side.

We shall briefly introduce an attempt below by further convert Conjecture 4.2.1 $(n = 3)$ to a matrix inequality problem. Denote by $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ a vector in $\mathbb{C}^2 \otimes \mathbb{C}^9$ where $\xi_1, \xi_2 \in \mathbb{C}^9$. $P_\xi = \begin{bmatrix} \xi_1\xi_1^* & \xi_1\xi_2^* \\ \xi_2\xi_1^* & \xi_2\xi_2^* \end{bmatrix}$ is the projection associated with $|\xi\rangle$. By Definition 1.1.2 it suffices to prove the following matrix inequality for any ξ.

$$[id_2 \otimes (\tau_3 \circ \psi_3)^{\otimes 2}]P_\xi = \begin{bmatrix} (\tau_3 \circ \psi_3)^{\otimes 2}(\xi_1\xi_1^*) & (\tau_3 \circ \psi_3)^{\otimes 2}(\xi_1\xi_2^*) \\ (\tau_3 \circ \psi_3)^{\otimes 2}(\xi_2\xi_1^*) & (\tau_3 \circ \psi_3)^{\otimes 2}(\xi_2\xi_2^*) \end{bmatrix} \geq 0 \tag{4.6}$$

Lemma 2.2.3 implies that the positivity of the above matrix is equivalent to three conditions, i.e.,

(a) $(\tau_3 \circ \psi_3)^{\otimes 2}(\xi_1\xi_1^*) \geq 0$,

(b) $\mathcal{R}((\tau_3 \circ \psi_3)^{\otimes 2}(\xi_1\xi_2^*)) \subseteq \mathcal{R}((\tau_3 \circ \psi_3)^{\otimes 2}(\xi_1\xi_1^*))$,

(c) $(\tau_3 \circ \psi_3)^{\otimes 2}(\xi_2\xi_2^*) \geq [(\tau_3 \circ \psi_3)^{\otimes 2}(\xi_2\xi_1^*)] \cdot [(\tau_3 \circ \psi_3)^{\otimes 2}(\xi_1\xi_1^*)]^{-} \cdot [(\tau_3 \circ \psi_3)^{\otimes 2}(\xi_1\xi_2^*)]$.

The condition (a) follows from Lemma 4.2.1 and the fact that ϕ_3 is completely positive. The condition (b) can be proved. Meanwhile, proving the condition (c) seems hard.

Bibliography

[AuSz15] Guillaume Aubrun and Stanishław J. Szarek, *Two proofs of Stømer's theorem*, arXiv:1512.03293.

[AuSz16] Guillaume Aubrun and Stanishław J. Szarek, *Dvoretzky's theorem and the complexity of entanglement detection*, arXiv:1510.00578.

[ASY13] Guillaume Aubrun, Stanishław J. Szarek and Deping Ye, *Entanglement thresholds for random induced states*, Communications on pure and applied mathematics, **67**(1) (2013), 129–171 .

[BDMSST99] C. H. Bennett, D. P. Dinvincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, *Unextendible Product Bases and Bound Entanglement*, Phys. Rev. L., **82** (1999), 5385.

[BDSW96] C. H. Bennett, D. P. Dinvincenzo, J. A. Smolin, and W. K. Wootters, *Mixed-state entanglement and quantum error correction*, Phys. Rev. A, **54** (1996), 3824-3851.

[BeGr03] A. Ben-Israel, Thomas N.E. Greville, *Generalized Inverses: Theory and Applications*, CMS Books in Mathematics, 2003.

[Bh07] R.Bhatia, *Positive Definite Matrices*, Princeton Series in Applied Mathematics, 2007.

[CHLW14] J. Cadney, M. Huber, N. Linden, and A. Winter, *Inequalities for the ranks of multipartite quantum states*, Linear Algebra and its Applications, **91**(3) (2015), 032308.

[CCMV11] L. Chen and E. Chitambar, K. Modi, and G. Vacanti, *Detecting multipartite classical states and their resemblances*, Phys. Rev. A, **83** (2014), 020101(R).

[ChDj11] L. Chen and D. Ž. Djoković, *Description of rank four entangled states of two qutrits having positive partial transpose*, J. Math. Phys., **52** (2011), 122203.

[ChDj11jpa] L. Chen and D. Ž. Djoković, *Distillability and PPT entanglement of low-rank quantum states*, J. Phys. A: Math. Theor., **44** (2011), 285303.

[ChDj16pra] L. Chen and D. Ž. Djoković, *Distillability of non-positive-partial-transpose bipartite quantum states of rank four*, Phys. Rev. A, **94**(5) (2016), 052318.

[ChDj13] L. Chen and D. Ž. Djoković, *Separability problem for multipartite states of rank at most 4*, J. Phys. A: Math. Theor., **46** (2013), 275304.

[ChDj16] L. Chen and D. Ž. Djoković, *Distillability of non-positive-partial-transpose bipartite quantum states of rank four*, Phys. Rev. A, **94**(5) (2016), 052318.

[ChLi14] L. Chen and Yu Li, *Nonlocal and controlled unitary operators of Schmidt rank three*, Phys. Rev. A, **89**(6) (2014), 062326.

[ChLi14ap] L. Chen and Yu Li, *On the Schmidt-rank-three bipartite and multipartite unitary operator*, Annals of Physics, **351** (2014), 682–703.

[ChLi15] L. Chen and Yu Li, *Decomposition of bipartite and multipartite unitary gates into the product of controlled unitary gates*, Phys. Rev. A, **452**(1) (2014), 153–171.

[LYT16] Lin Chen, Yu Yang and Wai-Shing Tang, *Schmidt number of bipartite and multipartite states under local projections*, Quantum Information Processing (2017), 16:75.

[CLMOW14] Eric Chitambar, Debbie Leung, Laura Mančinska, Maris Ozols and Andreas Winter,, *Everything you always wanted to know about LOCC*, Communications in Mathematical Physics, **328** (2014), 303–326.

[CKL92] Sung Je Cho, Seung-Hyeok Kye and Sa Ge Lee, *Generalized Choi maps in three-dimensional matrix algebra*, Linear Algebra and its Applications, **171** (1992), 213–224.

[Ch72] M.-D. Choi, *Positive linear maps on C^*-algebras*, Canad. Math. J., **24** (1972), 520–529.

[Ch75] M.-D. Choi, *Positive semidefinite biquadratic forms*, Linear. Alg. Appl, **12** (1975), 95–100.

[ChKo09] D. Chrúscínski and A. Kossakowski, *Spectral conditions for positive maps*, Commun. Math. Phys., **290**(3) (2009), 1051–1064.

[Cl06] L. Clarisse, *Construction of bound entangled edge states with special ranks*, Phys. Lett. A, **359** (2006), 603–607.

[Co97] J. B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, 1997.

[CMW08] Toby Cubitt and Ashley Montanaro and Andreas Winter, *On the dimension of subspaces with bounded Schmidt rank*, J. Math. Phys., **49** (2008), 2022107.

[DSSTT00] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, *Evidence for Bound Entangled States with Negative Partial Transpose*, Phys. Rev. A, **61** (2000), 062312.

[DVC00] W. Dür, G. Vidal and J. I. Cirac, *Three qubits can be entangled in two inequivalent ways*, Phys. Rev. A, **62** (2000), 062314.

[EiBr01] Jens Eisert and Hans J. Briegel, *Schmidt measure as a tool for quantifying multiparticle entanglement*, Phys. Rev. A, **64** (2001), 022306.

[EoKy00] M.-H. Eom and S.-H. Kye, *Duality for positive linear maps in matrix algebras*, Math. Scan., **86** (2000), 130–142.

[Gh10] Sevag Gharibian, *Strong NP-hardness of the Quantum Separability Problem*, Quantum Info. Comput., **10** (2010), 343–360.

[GMS15] V. P. Gupta, P. Mandayam and V. S. Sunder, *The Functional Analysis of Quantum Information Theory-A Collection of Notes Based on Lectures by Gilles Pisier, K.R. Parthasarathy, Vern Paulsen and Andreas Winter*, Lecture Notes in Physics, **902** (2015).

[Gu03] Leonid Gurvits, *Classical deterministic complexity of Edmond's problem and quantum entanglement*, Proceedings of the Thirty-fifth Annual ACM Symposium on Theory of Computing., STOC **03** (2003), 10–19.

[Ha07] Kil-Chan Ha, *Comment on: "Construction of bound entangled edge states with special ranks"*, Phys. Lett. A, **361** (2007), 515–519.

[Ha98] Kil-Chan Ha, *Atomic Positive Linear Maps in Matrix Algebras*, Publ RIMS, Kyoto Univ., **34** (1998), 591–599.

[HaKy16] Kil-Chan Ha and Seung-Hyeok Kye, *Construction of exposed indecomposable positive linear maps between matrix algebras*, Linear and Multilinear Algebra, **64**(11) (2016), 2188–2198.

[HaKy16jmp] Kil-Chan Ha and Seung-Hyeok Kye, *Various notions of positivity for bilinear maps and applications to tri-partite entanglement*, J. Math. Phys., **57** (2016), 015205.

[HaCh11] M. Hayashi and L. Chen, *Weaker entanglement between two parties guarantees stronger entanglement with a third party*, Phys. Rev. A, **84** (2011), 012325.

[HoHo99] Michał Horodecki and Paweł Horodecki, *Reduction criterion of separability and limits for a class of distillation protocols*, Phys. Rev. A, **59** (1999), 4206.

[HHH96] Michał Horodecki, Paweł Horodecki and Ryszard. Horodecki, *Separability of mixed states: necessary and suffcient condition*, Phys. Lett. A, **223** (1996), 1–8.

[HHH98] Michał Horodecki, Paweł Horodecki and Ryszard Horodecki, *Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?*, Phys. Rev. Lett., **80** 5239 (1998).

[HHH99] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki and Karol Horodecki, *Quantum entanglement*, Reviews of Modern Physics., **81** (2009).

[HLVC00] P. Horodecki, M. Lewenstein, G. Vidal and I. Cirac, *Operational criterion and constructive checks for the separability of low rank density matrices*, Phys. Rev. A, **62** (2000), 032310.

[HSTT03] P. Horodecki, J. A. Smolin, B. M. Terhal and A. V. Thapliyal, *Rank two bipartite bound entangled states do not exist*, Theoret. Comput. Sci., **292** (1999), 589–596.

[HJW93] L. P. Hughston, R. Jozsa, and W. K. Wootters, *A complete classification of quantum ensembles having a given density matrix*, Phys. Lett. A, **183** (1993), 14–18.

[It86] T. Itoh, *Positive maps and cones in C^*-algebras*, Math. Jpn., **31** (1986), 607–616.

[Ka51] R. V. Kadison, *Isometries of operator algebras*, Ann. of Math., **54**(2) (1951), 325–338.

[Ka52] R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algbebras*, Ann. of Math., **56**(2) (1952), 494–503.

[KKL11] Y.-H. Kiem, Seung-Hyeok Kye and J. Lee, *Existence of product vectors and their partial conjugates in a pair of spaces*, J. Math. Phys., **52** (2011), 122201.

[KiKy94] H.-J. Kim and Seung-Hyeok Kye, *Indecomposable positive linear maps in matrix algebras*, Bull. Lond. Math. Soc., **26** (1994), 575–581.

[KCKL00] B. Kraus, J. I. Cirac, S. Karnas and M. Lewenstein, *Separability in 2xN composite quantum systems*, Phys. Rev. A, **61** (2000), 062302.

[Ky13] Seung-Hyeok Kye, *Facial structures for various notions of positivity and applications to the theory of entanglement*, Rev. Math. Phys., **25** (2013), 1330002.

[KyOs12] Seung-Hyeok Kye and H. Osaka, *Classification of bi-qutrit positive partial transpose entangled edge states by their ranks*, J. Math. Phys., **53** (2012), 052201.

[Ma10] M. Marciniak, *On extremal positive maps acting between type I factors*, Noncommutative Harmonic Analysis with Applications to probability II, Banach Center Publications, **89** (2010), 201-221.

[MiOl15] M. Miller and R. Olkiewicz, *Topology of the cone of positive maps on qubit systems*, J. Phys. A: Math and Theor., **48**(25) (2015), 255203.

[NC10] Michael A. Nielsen and Issac L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, 2010.

[Pe96] Asher Peres, *Separability criterion for density matrices*, Phys. Rev. Lett., **77** (1996), 1413.

[Ro85] A. G. Robertson, *Positive Projections on C^*-algebras and an extremal positive map*, J. London. Math. Soc., **32**(2) (1985), 133 – 140.

[SBL01] Anna Sanpera, Dagmar Bruß and Maciej Lewenstein, *Schmidt-number witnesses and bound entanglement*, Phy. Rev. A., **63** (2001), 050301(R).

[Sh97] Peter W. Shor, *Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer*, SIAM Journal on Computing., **26**(5) (1997), 1484 – 1509.

[Sk11] Łukasz Skowronek, *Three-by-three bound entanglement with general unextendible product bases*, J. Math. Phys., **52** (2011), 122202.

[SkSt12] Łukasz Skowronek and Erling. Størmer, *Choi matrices, norms and entanglement associated with positive maps on matrix algebras*, J. Funct. Anal., **262** (2012), 639 – 647.

[SSZ09] Łukasz Skowronek, Erling Størmer and Karol Życzkowsk, *Cones of positive maps and their duality relations*, J. Math. Phys., **50** (2009), 050301(R).

[Sti55] W. F. Stinespring, *Positive functions on C^*-algebras*, Proc. Am. Math. Soc., **6** (1955), 211 –216 .

[St63] Erling Størmer, *Positive Linear maps of Operator Algebras*, Acta Mathematica, **110**(1) (1963), 233 –278 .

[St86] Erling Størmer, *Extension of positive maps into $B(\mathcal{H})$*, J. of Funct. Anal., **66** (1986), 235 –254 .

[St09] Erling Størmer, *Duality of cones of positive maps*, Münster J. of Math., **2** (2009), 299–310.

[St13] Erling Størmer, *Positive Linear Maps of Operator Algebras*, Springer Monographs in Mathematics, 2013.

[St16jmp] Erling Størmer, *Tensor powers of 2-positive maps*, Journal of Mathematical Physics, **51**(1) (2010), 102203.

[St16] Erling Størmer, *A decomposition theorem for positive maps, and the projection onto a spin factor*, Mathematica Scandinavica., **118**(1) (2016), 106–118.

[Te00] Barbara M. Terhal, *Bell inequalities and the separability criterion*, Phys. Lett. A., **271**(6) (2000), 319–326.

[TeHo00] Barbara M. Terhal and Paweł Horodecki, *Schmidt number for density matrices*, Phys. Rev. A., **61** (2000), 040301.

[To57] J. Tomiyama, *On the projection of norm one in W^*-algebras*, Proc. Jpn. Acad., **33** (1957), 608–612.

[To85] J. Tomiyama, *On the geometry of positive maps in matrix algebras. II*, Lin. Alg. Appl., **69** (1985), 169–177.

[ViDo06] Reinaldo O. Vianna and Andrew C. Doherty, *Distillability of Werner states using entanglement witnesses and robust semidefinite programs*, Phys. Rev. A, **74** (2006), 052306.

[Wa04] John Watrous, *Many Copies May Be Required for Entanglement Distillation*, Phys. Rev. Lett., **93** (2004), 010502.

[We89] R. F. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A, **40** (1989), 4277–4281.

[Wo76] S. L. Woronowicz, *Positive maps of low dimensional matrix algebras*, Rep. Math. Phys., **10**(2) (1976), 165–183.

[YDT16] Yu Yang, Denny H. Leung and Wai-Shing Tang, *All 2-positive linear maps from $M_3(\mathbb{C})$ to $M_3(\mathbb{C})$ are decomposable*, Linear Algebra and its Applications, **503** (2016), 233–247.

[Ye10] Deping Ye, *On the comparison of volumes of quantum states*, J. Phys. A: Math. Theor., **43** (2010), 315301.

[ZCH10] Huangun Zhu, Lin Chen and Masahito Hayashi, *Additivity and non-additivity of multipartite entanglement measures*, New Journal of Physics, **12** (2010), 083002.

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